Numerical Techniques for a Two-Dimensional Kinematic Overland Flow Model

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Abstract

The assumptions behind the kinematic equations of flow are presented together with a description of the applicability of the equations to overland flow problems. Numerical methods of solution of the one-dimensional and two-dimensional equations are compared before selecting a backward-central explicit finite difference scheme for a two-dimensional model. Efficiency and stability of the numerical scheme is discussed and comparison with results from an experimental catchment verifies the model. The model is suited for analysis of sheet flow off two-dimensional planes such as encountered in natural catchments. Typical output of the program in the form of graphical contour plots, three-dimensional views, rainfall variation and hyetographs are presented.

Introduction

Existing means for estimating floods from specific catchments are limited and based on simplified assumptions. Topography and catchment surface characteristics are not accounted for in unit hydrograph methods. Neither is catchment shape, infiltration variation or the effect of urbanization. Storm patterns cannot be accounted for and the assumption of a rectangular hyetograph over the entire catchment is often dangerous. The effect of infiltration and storm variation in time and space can be accounted for in the numerical model outlined in this paper. Storms off abnormally steep or distorted catchments can be severe and unpredictable unless the catchment is modelled. Previous modelling tools have been formidable. The differential equations of motion would be extremely costly to solve numerically. Alternative empirical methods such as Muskingum routing methods are simplistic and inaccurate. It is hoped that the development of the present computer program will lead to acceptance of models as the only accurate and justified means of assessing catchment storm runoff.

Basic Equations

The one-dimensional Kinematic equations are derived by simplifying the Dynamic equations for the one-dimensional case commonly known as the St. Venant equations. Derivations of the St. Venant equations are obtained by applying the principles of conservation of mass and momentum eg. Chow (1959), Yen (1973).

The St. Venant equations are:

continuity $$\frac{\partial y}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (1)$$

momentum $$g\frac{\partial y}{\partial x} + \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = g(S_o - S_f) \qquad (2)$$

where

- x : space axis along the direction of flow (m)
- y : depth of water above the bed (m)
- t : time (s)
- q : average discharge across a section per unit width (m^2/s)
- v : average velocity across a section (m/s)
- S_o : bed slope
- S_f : friction slope

The Kinematic continuity equation is identical to the corresponding St. Venant equation with the addition of lateral inflow while the dynamic equation in the case of the Kinematic equations is obtained by assuming that the bed slope is equal to the friction slope. The one-dimensional Kinematic equations thus become:

$$\frac{\partial y}{\partial t} + \frac{\partial q}{\partial x} = i_e \qquad (3)$$

$$S_o = S_f \qquad (4)$$

where

i_e = excess rainfall (m/s)

Equation (4) is the assumption for steady uniform flow. A major simplification in the dynamic equations is thus possible. Due to this and the assumptions in deriving the dynamic equation from the general Navier-Stokes equations (Yen, 1973) there are limitations on the applicability of the Kinematic equations. The assumptions made in reducing the general equations of motion as derived by Yen (1973), to the St. Venant ones are: The flow must be gradually varied i.e. no rapid changes in the flow cross sectional area, implying that fluid acceleration in all directions other than the direction of flow is negligible; the bed slope must be small i.e. $\Theta \simeq \sin\Theta \simeq \tan\Theta$; velocity is uniformly distributed over any cross-sectional area; pressure distribution along a cross-section is hydrostatic; lateral inflow or outflow carries no momentum; and in order to reduce the St. Venant equations to the Kinematic equations it is necessary to neglect the effect of space and time variation of velocity and depth in St. Venant's equation of motion. These assumptions were shown to be satisfactory for overland flow in most cases (Lighthill and Whitham, 1955; Overton and Meadows, 1976).

By assuming that the bed slope equals the friction slope and by using any existing open-channel flow friction equation we can express the discharge at any point and time as a function of the water depth only as follows

$$q = \alpha y^m \qquad (5)$$

where α and m are parameters depending on the open-channel flow friction equation used.

For example, for the Darcy-Weisbach equation:

$$\alpha = (8gS_o/f)^{1/2} \qquad (6)$$

$$m = \frac{3}{2} \qquad (7)$$

for turbulent flow

where

g = acceleration due to gravity (m/s^2)

f = Darcy-Weisbach friction coefficient

Unfortunately f is a function of the hydraulic radius, and for low Reynolds numbers it is also a function of flow velocity and fluid viscosity.

For the Manning-Strickler equation applicable to turbulent flow with a rough boundary

$$\alpha = 7(g\,S_o)^{1/2}/(k)^{1/6} \qquad (8)$$

$$m = \frac{5}{3} \qquad (9)$$

where

k = equivalent to Nikuradse's roughness.

Note that values of the constant in equation (8) have been quoted by various authors as varying between 6,7 (Jaeger, 1956) and 7,7 (Einstein and Barbarossa, 1952). The Manning-Strickler equation is preferred to the Darcy equation as the friction coefficient is a unique function of the surface. Both the Darcy-Weisbach and the Manning-Strickler equations are intended fo turbulent flow conditions. The transition between laminar and turbulent flow is shown to be around Reynolds number of 500 (Woolhiser, 1977). where Reynolds number is defined as:

$$Re = \frac{vR}{\nu} \qquad (10)$$

where

v = average velocity across a section (m/s)
ν = kinematic viscosity of water = $1{,}33 \times 10^{-6} m^2/s$
R = hydraulic radius of section

For Reynolds number less than 500 laminar flow persists and the Darcy-Weisbach equation is applicable. The resistance coefficient is a function of Reynolds number as follows:

$$f = \frac{K}{Re} \qquad (11)$$

where K is a function of surface configuration and rainfall intensity (Woolhiser, 1977).

For turbulent flow Shen and Li (1973) also discussed the effect of rainfall on the friction coefficients used in the channel formula. Using the Darcy-Weisbach friction formula for Reynolds numbers greater than 2000 Shen and Li found that rainfall affected the friction coefficient by no more than 5%.

For the purposes of this model the Manning-Strickler equation as well as the Darcy-Weisbach equation were used as options for turbulent flow.

Two-dimensional kinematic equations

The kinematic equations can be written for the two-dimensional case as follows. The continuity equation becomes

$$\frac{\partial y}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = i_e \qquad (12)$$

where

q_x is the flow in the x direction (m^2/s)
q_z is the flow in the z direction (m^2/s)

A proof of this equation can be found in Dronkers (1964). For two-dimensional flow two equations of motion are required. The relevant Kinematic equations become

$$S_{ox} = S_{fx} \qquad (13)$$

$$S_{oz} = S_{fz} \qquad (14)$$

where

S_{ox} = bed slope in x direction
S_{oz} = bed slope in z direction
S_{fx} = friction slope in x direction
S_{fz} = friction slope in z direction

By considering velocity as a vector which can be resolved in two perpendicular directions one can extend the one-dimensional flow equation (5) into two-dimensional flow. By squaring both sides of equation (5) one obtains:

$$q_x = \frac{1}{q_t}(\alpha_x y^m)^2 \qquad (15)$$

$$q_z = \frac{1}{q_t}(\alpha_z y^m)^2 \qquad (16)$$

where

$$q_t = (q_x^2 + q_z^2)^{1/2} \qquad (17)$$

and

α_x = α as a function of S_{ox}
α_z = α as a function of S_{oz}

This idea for two dimensional flow was used by Orlob (1972). It will be noticed that q_t is always positive while q_x and q_z can be positive or negative as $(\alpha_x)^2$ and $(\alpha_z)^2$ are functions of S_{ox}

and S_{oz} respectively. By using Manning-Strickler equations (8) and (9) one can obtain:

$$q_x = \frac{1}{q_t} \frac{49g\, S_{ox}}{(k)^{1/3}} y^{10/3} \qquad (18)$$

$$q_z = \frac{1}{q_t} \frac{49g\, S_{oz}}{(k)^{1/3}} y^{10/3} \qquad (19)$$

where q_t is defined in equation (17).

Solution of equations

The present consideration lies in the solution of a set of non-linear partial differential equations. Since analytical solutions are not available numerical solutions are required. Considerable literature exists on the solution of the St. Venant equations with methods ranging from explicit finite difference schemes to finite element methods.

The following condenses the basic thoughts behind the methods of solution of the equations. An explicit finite difference scheme is then selected as it is known to be economical and shown to be stable. The characteristic equations are considerred first as they will demonstrate certain basic principles. By using the method of characteristics one can show for the Kinematic equations that:

$$\frac{dq}{dy} = \frac{dx}{dt} \qquad (20)$$

and by assuming

$$\frac{dx}{dt} = \alpha m y^{m-1} \qquad (21)$$

it can be shown that

$$\frac{dq}{dx} = i_e \qquad (22)$$

$$\frac{dy}{dt} = i_e \qquad (23)$$

$$\frac{dq}{dt} = \left(\frac{dx}{dt}\right) i_e \qquad (24)$$

$$\frac{dy}{dx} = i_e/(dx/dt) \qquad (25)$$

Proof of this can be found in Overton and Meadows (1976). Similarly for the St. Venant equations it was shown (Henderson, 1966; Abbott, 1979) by assuming

$$\frac{dx}{dt} = (v \pm \sqrt{gy}) \qquad (26)$$

that

$$\frac{d(v \pm 2\sqrt{gy})}{dt} = (S_o - S_f)g - \frac{q_L v}{A} \qquad (27)$$

where

q_L = lateral inflow per unit length along the flow (m^2/s)
A = cross-sectional area of channel (m^2)

Equations (20) to (25) are called the characteristic Kinematic equations while equations (26) and (27) are called the characteristic dynamic equations.

By using the method of characteristics on the St. Venant equations it can be shown that a disturbance described by the unsteady dynamic flow equations can be propagated with a velocity $v \pm \sqrt{gy}$ relative to the ground or $\pm\sqrt{gy}$ relative to the flow. Since the dynamic equations have often been used to describe unsteady flow caused by a disturbance in an originally steady flow this disturbance is seen in the form of a wave called a dynamic wave. With the use of the Kinematic equations, however, the existence of a kinematic wave is not always obvious, as in the case of the rainfall runoff relationship. The meaning of the characteristic Kinematic equations is different for this case.

For example consider a point in space in the rainfall-runoff situation. The water leaving the point is governed by the open-channel friction formulae (i.e. equation (5)) and is usually not equal to the water entering it from upstream points and from the excess rainfall. To satisfy continuity the water depth at the point changes. All that the characteristic equations now mean is that information about this abnormality in flow will be propagated downstream at a speed of dx/dt relative to the ground, given by equation (21).

Since this happens at most points most of the time in a normal catchment one cannot talk of travelling waves but of continuity satisfying itself at all points all the time. These considerations are important in the solution of the equations, particularly in the choice of a difference scheme and in defining stability and accuracy criteria for the selected scheme.

Explicit finite difference schemes

Explicit finite difference schemes have been found by Ligget and Woolhiser (1967) and Strelkoff (1970) to be economical and efficient. The major problems accompanying them however are those of stability and accuracy. An explicit finite difference scheme was used in the model and was shown to be stable and accurate enough. In the analysis the idea of propagating information along characteristics from the Kinematic equations is used and some possible explicit finite difference schemes are compared on this basis. Considerable effort was spent in seeking an economical yet accurate and stable numerical scheme for the solution of the two-dimensional equations. In fact many schemes were attempted which proved unstable to varying degrees. The scheme that finally satisfied accuracy and stability criteria is a scheme that propagates information in a way similar to that suggested by the Kinematic characteristic equations (21) to (25). Before this is done the basic concepts of stability and accuracy are summarised as they will be used in the analysis to follow.

Accuracy and stability of explicit difference schemes

General

The relevant terms will be defined in order to facilitate understanding of the interrelationship between stability and accuracy. Many natural systems which are continuous functions can be described by differential equations. If differential equations cannot be solved mathematically one can resort to numerical methods by approximating the differential equations with difference schemes. One usually assumes that the differential equations represent the system exactly. This will be assumed in the analysis to follow despite the fact that it has been noticed that this is not necessarily the case and that a difference scheme considerably different from the differential equations used to describe a system, yields more accurate results than a difference scheme similar to the differential equations when compared with experimental results eg. Abbott (1974).

The difference between the differential equations and the difference scheme approximating them is called a *Truncation error* (Tr.)

i.e. Differential Equations = Difference Scheme + Tr (28)

The truncation error can be easily established using Taylor's expansion. There is also a difference in the solutions of the two schemes which one calls the Error (E)

i.e. Solution of Differential Equations = Solution of Difference Equations + E (29)

The exact value of the Error cannot really be obtained in our case as we are unable to obtain the solution of the differential equation. We say that a difference scheme is *consistent* with a set of Differential Equations if the Truncation error tends to zero as the space and time increments tend to zero.

i.e. Consistent if limit Tr = 0

$$\triangle x, \triangle t \rightarrow 0 \qquad (30)$$

We say that the solution of the difference scheme is *convergent* with the solution of the differential equations if the Error tends to zero as the space and time increments tend to zero.

i.e. Convergent if limit E = 0

$$\triangle x, \triangle t \rightarrow 0 \qquad (31)$$

Numerical diffusion is the process in which the Error (E) is formed. It is the development of the truncation error (Tr) to the error (E) through the numerical technique used.

There are theories as to when convergence exists e.g. Lax's (1954) theory, proved by Richtmyer and Morton (1967) which states that for linear equations with constant coefficients operating on uniformly continuous initial and boundary data the following theorem holds. Given a properly posed initial-value problem and a finite difference approximation to it that satisfies the consistency conditions, stability is the necessary and sufficient condition for convergence. This is however proved only for linear equations and also according to Abbott (1979) it breaks down when there are discontinuities in flow.

Stability and accuracy criteria for an explicit difference scheme

Since one is dealing with non-linear partial differential equations (p.d.e's) there is no rigorous proof specifying stability criteria. For linear p.d.e's, however, stability analyses exist. Von Neuman (1949) was first to devise a powerful technique for determining stability criteria for linear p.d.e's. This and other similar techniques are useful in understanding a few basic principles of stability but are outside the scope of this paper.

It was found that both stability and accuracy are influenced by the values chosen for the space increment $\triangle x$ and the time increment $\triangle t$. We can best represent the effect of $\triangle x$ and $\triangle t$ values on stability and accuracy in Fig. 1.

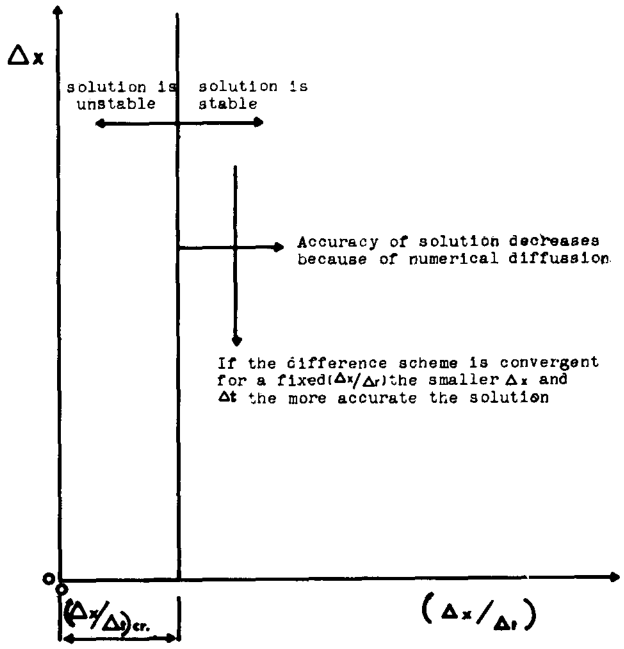


Figure 1
Effect of values of $\triangle x$ and $\triangle t$ on stability and accuracy

In Fig. 1 $\left(\frac{\triangle x}{\triangle t}\right)_{cr}$ represents a critical $\frac{\triangle x}{\triangle t}$ ratio to be determined later.

The main criteria therefore in the selection of $\triangle x$ and $\triangle t$ values for a difference scheme are:

a) that the scheme shall proceed under stable conditions

i.e. $\frac{\triangle x}{\triangle t} \geq \left(\frac{\triangle x}{\triangle t}\right)_{cr}$

b) have $\frac{\triangle x}{\triangle t}$ as close to $\left(\frac{\triangle x}{\triangle t}\right)_{cr}$ to minimize diffusion errors and obtain optimal accuracy.

c) make sure that the difference scheme is convergient by running the scheme with different $\triangle x$'s and $\triangle t$'s and comparing with analytical results in a simple case. Then decrease the values of $\triangle x$ and $\triangle t$ to improve accuracy bearing in mind that as this is done one increases computer time and core storage and thus running costs.

Determining $\left(\frac{\triangle x}{\triangle t}\right)_{cr}$

$\left(\frac{\triangle x}{\triangle t}\right)_{cr}$ has been shown to be the speed of the wave, disturbance or information as it is propagated. The principle of the above criterion is illustrated in Fig. 2. In diagram (a) information within the range i − 1, i + 1 is required by the true propagated speeds while in diagram (b) information outside i − 1, i + 1 range is required. Since information outside this range is not propagated by the numerical scheme it cannot be found and

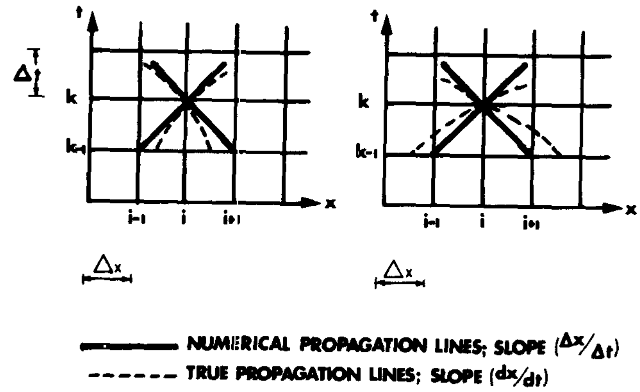


Figure 2
Comparison of numerical and theoretical propagation of information

thus instability will result. (For further explanations see Stoker, (1957); Henderson, (1966)).

Therefore for stability

$$\frac{\Delta x}{\Delta t} \geq \frac{dx}{dt} \qquad (32)$$

This is referred to as the "CFL condition" after Courant et al (1928).

However it has been noticed that even if we satisfy the CFL condition it is not necessarily true that the solution of our difference scheme is stable (eg. by Lax, 1954; Richtmyer and Morton, 1967; Abbott, 1974). There are two possibilities for this to happen. There could be a physical discontinuity in the flow, eg. a bore or a hydraulic jump.

In terms of characteristics this implies the intersection of two or more characteristics. Theoretically this results in different values of flow properties for a fixed place and time. In a difference scheme with a fixed grid this theoretical multivaluedness

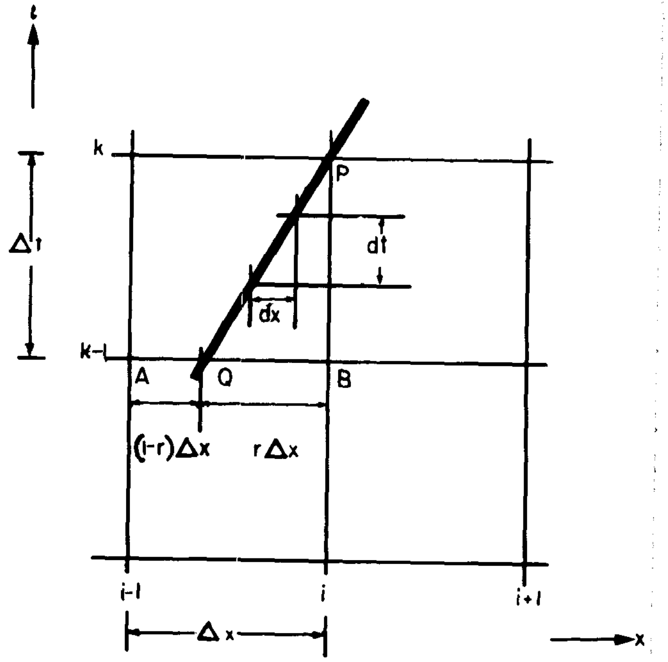


Figure 3
The principle of weighted averages for information propagation

does not exist and in the solution is present in the form of oscillations. If the difference scheme tends to amplify these oscillations instability will occur. If however these oscillations get damped stability will result and our scheme is referred to as a dissipative difference scheme.

The difference scheme being used can also cause oscillations called parasitic waves. It has been noticed (e.g. by Abbott, 1974) that the parasitic waves do not only occur when a physical discontinuity occurs but can arise out of the numericall procedure used. Therefore certain difference schemes have been found to produce parasitic waves while other do not, when considering the same physical problem.

There are two ways these problems can be overcome. If a physical discontinuity exists it can be located, the laws governing the discontinuity can be applied, and our laws governing continuous flow can be applied to each side. It is also possible to adjust any difference scheme to dampen instead of amplifying parasitic waves. The solutions obtained from these 'dissipative difference schemes', are called 'weak solutions', as in this way stability is obtained at the loss of accuracy (see Lax, 1954). Abbott (1974) describes the dissipative schemes and the amount of accuracy lost extensively.

If one considers the method of setting up a dissipative scheme, it will also illustrate the principle of the weighted averages at a time interval of flow properties to transfer to the next time interval.

Consider for example a difference scheme as shown in Fig. 3 and the way information about depth (y) is propagated. Depth at time $t = k-1$ is taken to be as

$\frac{1}{2}\left(y_{i-1}^{k-1} + y_{i}^{k-1}\right)$ (see Fig. 3). Suppose now one

wants to propagate the depth at point Q. Then interpolating linearly between points A and B one must use depth at Q at time $t = k-1$ as $(1-r)y_{i}^{k-1} + ry_{i-1}^{k-1}$.

If one uses the fact that information is truly propagated at a speed of dx/dt then the slope of line QP should be the value of dx/dt at point Q (representing a point in space at a particular time) denoted as $(dx/dt)_Q$.

Therefore strictly speaking the value of r should be

$$r = \left(\frac{dx}{dt}\right)_Q \Big/ \frac{\Delta x}{\Delta t} \qquad (33)$$

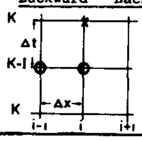
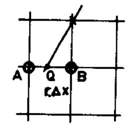
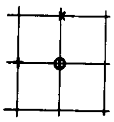
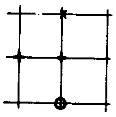
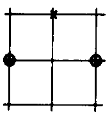
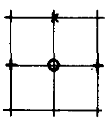
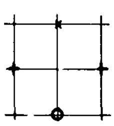
A dissipative difference scheme is one as described above but with r chosen in such a way as to dampen oscillations. The discrepancy between r chosen and r in equation (33) will result in loss of accuracy in the solution of the difference scheme.

Choosing an explicit difference scheme

The one dimensional Kinematic equations are employed to demonstrate the selected numerical scheme. Possible difference schemes are used to solve these equations in problems which can be solved with analytical methods. The analytical solutions are then compared with results from the numerical solutions. The difference schemes are evaluated on the basis of accuracy and stability. As more complicated problems are considered some of the difference schemes will be eliminated.

The different difference schemes are classified in terms of the way they propagate information; i.e. in the way $\partial q/\partial x$ and $\partial h/\partial t$ are defined, and are shown in Table 1. From the theory of

TABLE 1
DIFFERENT EXPLICIT DIFFERENCE SCHEMES

Difference Scheme	Discharge rate $\frac{dq}{dx}$ at t = k-1	Depth y at t = k-1 (or t = k-2)
1. Backward - Backward e.d.s. 	$\frac{(q_i^{k-1} - q_{i-1}^{k-1})}{\Delta x}$	$\frac{1}{2}(y_{i-1}^{k-1} + y_i^{k-1})$
2. Weighted-Average Backward - Backward e.d.s. 	$\frac{(q_i^{k-1} - q_{i-1}^{k-1})}{\Delta x}$	$r y_{i-1}^{k-1} + (1-r)\, y_i^{k-1}$
3. Backward - Central e.d.s. 	$\frac{(q_i^{k-1} - q_{i-1}^{k-1})}{\Delta x}$	y_i^{k-1}
4. Backward - Central - Leap frog e.d.s.* 	$\frac{(q_i^{k-1} - q_{i-1}^{k-1})}{\Delta x}$	y_i^{k-2}
5. Diffusive Scheme (Central) 	$\frac{(q_{i+1}^{k-1} - q_{i-1}^{k-1})}{2\Delta x}$	$\frac{1}{2}\ y_{i-1}^{k-1} + y_{i+1}^{k-1}$
6. Unstable Scheme (Central) 	$\frac{(q_{i+1}^{k-1} - q_{i-1}^{k-1})}{2\Delta x}$	y_i^{k-1}
7. Leap-frog Scheme (Central)* 	$\frac{(q_{i+1}^{k-1} - q_{i-1}^{k-1})}{2\Delta x}$	y_i^{k-2}

***first time step done by diffusive scheme**

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x point where flow properties are to be calculated
\+ points used for calculating discharge at time t = k−1
0 points used for calculating depth at time t = k−1
e.d.s. explicit difference scheme

$$r = \left(\frac{dx}{dt}\right)_Q \Big/ \left(\frac{\Delta x}{\Delta t}\right)$$

Kinematic equations only the first four schemes should be considered as they are the only ones which propagate information in the direction of flow as it happens in Kinematic theory. However the other schemes will also be considered for comparison.

In the numerical solutions for scheme 2, in the definition of r instead of using $(dx/dt)_Q$ one will use $(dx/dt)_B$. This is only done for simplicity and the effect it has on the results has been considered by using an iterative scheme to determine the position of Q. This is done by continuously adjusting y at Q. y is found by using a linear interpolation between the space intervals. The results obtained have no appreciable difference on the results where (dx/dt) at B is used.

Problem 1

Consider the plane shown in Fig. 4 with constant slope and roughness, of length L. Let i represent excess rainfall such that

$$\begin{aligned} i &= 0 && \text{for } t < 0 \\ &= i_e = \text{constant} && \text{for } 0 \geq t \geq t_r \\ &= 0 && t_r < 0 \end{aligned}$$

where t_r is the storm duration.

A complete analytical solution of this problem can be found in the literature, e.g. Woolhiser and Ligget (1967); Overton and Meadows (1976), and will not be repeated here. A complete solution consists of depth and discharge variation at any point with respect to time and depth profiles at all times.

Equilibrium state of catchment

Consider an arbitrary point on a plane. Water flowing towards the point leaves it at the same rate with the result that $\partial q/\partial x = 0$ and the depth (y) increases linearly with time (t). However at the origin (x = 0) water only leaves the point without any entering it from upstream. Because of this flow deficiency a disturbance is created at the origin propagating downstream with a velocity dx/dt. When this disturbance reaches the downstream side of the plane we will have a state of equilibrium i.e. all the water entering the plane will be leaving it. The time required for this

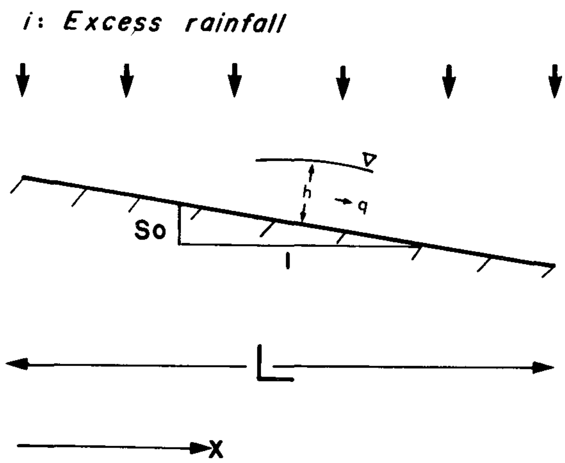

Figure 4
Problem 1: Plane with lateral inflow

to happen is the time of equilibrium of the catchment (t_e) and is given by

$$t_e = \left(\frac{L}{\alpha i_e^{m-1}} \right)^{1/m} \qquad (34)$$

The depth of the water at the time of equilibrium (y_e) is given by

$$y_e = i_e t_e \qquad (35)$$

and the discharge at the time of equilibrium (q_e) by

$$q_e = i_e L \qquad (36)$$

For derivations of these equations see Overton and Meadows (1976).

Numerical solutions of the problem using the different explicit difference methods

The Kinematic equations (3) and (5) together with the Manning-Strickler formula (8) and (9) were used. A constant slope of 1:10 and a Nikuradse's roughness of 0,2 mm were considered. The space and time intervals ($\triangle x$ and $\triangle t$ respectively) were chosen in such a way as to satisfy the CFL condition of stability i.e.

$$\frac{\triangle x}{\triangle t} \geq \frac{dx}{dt} = \alpha m (y_{max})^{m-1}$$

where y_{max} was taken to be the depth at the outlet at equilibrium from equation (35).

The storm duration was taken to be approximately double the time of equilibrium of the plane — obtained from equation (34) — and the programs were run for approximately four times the time to equilibrium.

Boundary conditions

There are two boundary conditions that can be used at the origin. One may assume that the depth at the origin is always zero and that all the water entering the origin leaves it in the form of discharge. This has been assumed in all existing theories. One must then define

$$\left(\frac{\partial q}{\partial x} \right)_1^k = i_e \qquad (37)$$

$$y_1^k = 0{,}0 \qquad (38)$$

therefore

$$q_1^k = i_e \triangle x/2 \qquad (39)$$

$\triangle x/2$ is used instead of $\triangle x$ since the origin space interval is only $\triangle x/2$. Programs using $\triangle x$ instead, showed no appreciable difference in their results.

One can also assume that the discharge at the origin is controlled by the depth of water at the origin as assumed for the rest of the points. For this case we must then use the same equations as with the other points. The effect of using the two different boundary conditions will be shown later.

Initial conditions

For the first time step it is assumed that the water depth at all points — except at the origin in the case of the first boundary condition — is given by

$$y_i^2 = i_e \triangle t \qquad (40)$$

Schemes were run under both boundary conditions. A subscript a is used to denote that the discharge at the origin is the same as the excess rainfall and the subscript b denotes that the discharge at the origin is a function of the water depth.

Analysis of results

The results from the numerical schemes were compared with the analytical results as follows. Graphs were drawn comparing depth and discharge variation with time at the outlet. Graphs were also drawn comparing depth variation along the catchment at different time intervals. The speed at which the disturbance was propagated in the numerical techniques was also compared to the speed at which the disturbance should have been propagated. This was done as follows:

The theoretical speed of the disturbance is given by the relationship $dx/dt = \alpha m y^{m-1}$. Along the path traced by this characteristic the relationship $dy/dx = i_e/\alpha m y^{m-1}$ holds (see equations (21) and (25).

Solving equation (25) one obtains

$$(\alpha m y^{m-1}) dy = i_e dx$$

Integrating: $\alpha(y-y_o)^{m-1} = i_e(x-x_o)$

At the origin when disturbance occurs $y_o = 0$ and $x_o = 0$

Thus one obtains $\alpha y^m = i_e x \qquad (41)$

As the disturbance travels it superimposes its height onto the water profile with the result that equation (41) describes the water profile at equilibrium. Numerically the water profile at equilibrium is also obtained. Regarding α and i_e as constant, a hypothetical value of m is obtained. When this is used in equation (21) i.e. $\frac{dx}{dt} = \alpha m y^{m-1}$ the numerical value of the speed of the disturbance is obtained. This is compared with the theoretical wave speed denoted by the subscript th.

By using equations (21) and (41) i.e.

$$\left(\frac{dx}{dt} \right)_{th} = \alpha m y^{m-1} = \alpha m \left(\frac{i_e x}{\alpha} \right)^{\frac{m-1}{m}}$$

$$\alpha m \frac{(i_e x)^{\frac{m-1}{m}}}{\alpha^{\frac{m-1}{m}}} = \alpha^{1/m} m \left(i_e x \right)^{\frac{m-1}{m}}$$

$$\left(\frac{dx}{dt} \right)_{th} = \alpha^{1/m} m \left(i_e x \right)^{\frac{m-1}{m}} \qquad (42)$$

Table (2) summarises the results of all the difference schemes.

TABLE 2
COMPARISON OF DISTURBANCE SPEEDS CAUSED BY DIFFERENT EXPLICIT DIFFERENCE SCHEMES WITH THEORETICAL DISTURBANCE SPEEDS

Difference Scheme	a: discharge at the origin equals excess rainfall							b: discharge at the origin is a function of depth						
Disturbance speed ratios at different space intervals	1a	2a	3a	4a	5a	6a	7a	1b	2b	3b	4b	5b	6b	7b
x = 0,0				RESULTS NOT AVAILABLE							RESULTS NOT AVAILABLE			RESULTS UNRELIABLE
0,2	3,08	1,41	0,89		1,39	0,89	1,10	1,50	1,18	0,89		1,12	0,89	
0,4	2,73	1,44	0,94		1,23	0,94	1,04	1,72	1,30	0,94		0,98	0,94	
0,6	2,50	1,43	0,96		1,17	0,96	1,02	1,78	1,34	0,96		0,95	0,96	
0,8	2,35	1,44	0,97		1,13	0,97	1,01	1,80	1,36	0,97		0,95	0,97	
1,0	2,24	1,44	0,98		1,11	0,99	1,01	1,80	1,38	0,98		0,95	0,98	
1,2	2,15	1,44	0,98		1,09	0,97	1,01	1,79	1,39	0,98		0,95	0,97	
1,4	2,08	1,44	0,98		1,09	0,97	1,00	1,77	1,39	0,98		0,96	0,98	
1,6	2,02	1,44	0,99		1,08	1,00	1,00	1,75	1,40	0,99		0,97	0,99	
1,8	1,97	1,44	0,99		1,07	0,99	1,00	1,74	1,40	0,99		0,97	0,99	
2,0	1,93	1,40	0,99		1,10	0,99	1,00	1,72	1,40	0,99		1,01	0,99	
Discharge at equilibrium of numerical scheme/theoretical	0,28	0,48	1,02		0,85	1,05	1,02	0,35	0,52	1,03		1,00	1,03	
Time of concentration numerical scheme/theoretical (approximate)	0,55	0,70	1,02		0,95	1,02	1,06	0,60	0,75	1,02		1,00	1,02	
Stability of Results	S	S	S	U	S	SU	OS	S	S	S	U	OS	SU	OS

Index for stability

S = stable solution
U = unstable solution
Su = unstable solution — however oscillations do not vary too much
OS = oscillations inbetween successive points i.e. two solutions exist made out of each alternate point

Discussion of the results

Stability

Some of the schemes were found to be stable and some unstable, the reason being the formation of parasitic waves in the numerical techniques used. The use of the leap-frog method (i.e. using information from 2 time steps before) in the backward difference schemes results in total instability i.e. schemes 4a and 4b. The use of the leap-frog method in central difference schemes, i.e. 7, results in point to point oscillation solutions oscillating from central value by an amount of ±4% in 7a and about 100% in 7b. Scheme 6 was found to be unstable as well; oscillations were not very big but the solution was still unstable. This scheme is commonly known as the unstable scheme as it was shown to be unstable when used with the dynamic equations (Ligget and Woolhiser, 1967). Scheme 5 is commonly known as the diffusive scheme. Its stability appears to depend on the boundary conditions. Using 'a' boundary conditions the scheme is stable but while using 'b' boundary conditions oscillations are noticed from point to point.

Even if some schemes are only slightly unstable they should be eliminated as they cannot be trusted especially for more complicated topography. Therefore one must eliminate schemes 4a, 4b, 6a, 6b, 5b, 7a and 7b.

Accuracy

Apart from schemes 1 and 2 the remainder of the schemes are very accurate. Their deviation from the correct solution is more due to diffusion errors which can be minimised by choosing a $\Delta x/\Delta t$ ratio closer to $(dx/dt)_{max}$.

The effect of using a weighted average of flow properties at a time interval is shown by comparing the results of schemes 1a and 2a and schemes 1b and 2b. Observe that by using this technique the disturbance tends to travel at a constant speed. Note however that in the other schemes the weighted average technique does not apply.

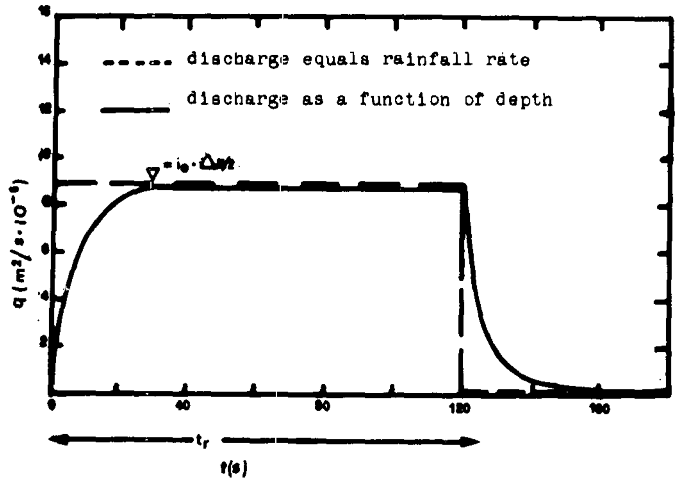

Figure 5
Problem 1: Discharge variation at the origin

Boundary conditions

Discharge at the origin varies according to the boundary condition used. A typical discharge versus time relationship at the origin is shown in Fig. 5. It can be seen that there is no appreciable difference when using either a or b boundary condition. It can also be seen from Table 2 by comparing a with b results that an improvement is noticed in the results by using condition b in schemes 1, 2 and 5 while no effect is seen in the other schemes.

Summarising

Schemes 3a, 3b and 5a were the only schemes that yielded acceptable results.

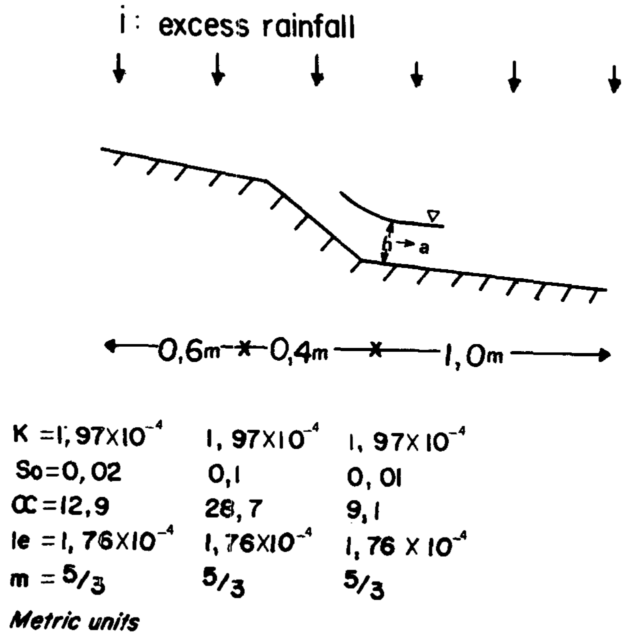

Figure 6
Problem 2: Cascade of planes

Problem 2

Cascade of planes

Consider three planes in sequence as shown in Fig. 6 with slopes, dimensions and roughness as shown.

As before let i represent excess rainfall such that

$$i = i_e \text{ for } 0 \leq t \leq tr$$

$$= 0 \text{ otherwise}$$

i_e is constant and evenly distributed over the catchment.

Existing theory for cascade of planes

Kibler and Woolhiser (1970) developed a Kinematic solution for a cascade of planes. The time of equilibrium was shown to be given by equation (43).

$$t_e \doteqdot \frac{Y_\infty}{i_e} \qquad (43)$$

where, when the Manning-Strickler formula is used, Y_∞ is given by:

$$Y_\infty = \sum_{i=1}^{p} \{(x_i^m - x_{i-1}^m) \times (\frac{i_e}{\alpha_i})^{1/m}\}\frac{1}{L} \qquad (44)$$

where subscript i refers to the ith plane
p is the number of planes
x_i is the distance of the end of the ith plane from the origin
L is the total length of the cascade
Y is the volume of water stored on the plane
Subscript ∞ refers to time infinity i.e. at equilibrium

The average depth of the water at the time of equilibrium is equal to Y_∞, while the discharge at equilibrium is given by equation (3) where α is the α of the last plane.

Applying these formulae to our problem:

$$Y_\infty = \frac{(1,76x10^{-4})^{0.6}}{2,0}\left(\frac{(0,6^{1.6})}{12,9^{0.6}} + \frac{(1,0^{1.6}-0,6^{1.6})}{28,7^{0.6}} + \frac{(2,0^{1.6}-1,0^{1.6})}{9,1^{0.6}}\right)$$

$$= 2,09 \times 10^{-3} m$$

$$t_e \doteqdot t_L = \frac{S_\infty}{i_e} = \frac{20,9 \times 10^{-4}}{1,76 \times 10^{-4}} = 11,9 \text{ s}$$

$$q_e = 9,1 \times (20,9 \times 10^{-4})^{1.6} = 3,1 \times 10^{-4} \text{ m}^2/\text{s}.$$

where t_L is the lagtime of the catchment as defined by Overton and Meadows (1976).

Numerical solutions

Numerical solutions were obtained using schemes 3a, 3b and 5a. Δx and Δt were chosen as described before to satisfy the CFL condition. The results and the theoretical solution are shown in Fig. 7. It can be seen that schemes 3a and 3b yielded identical

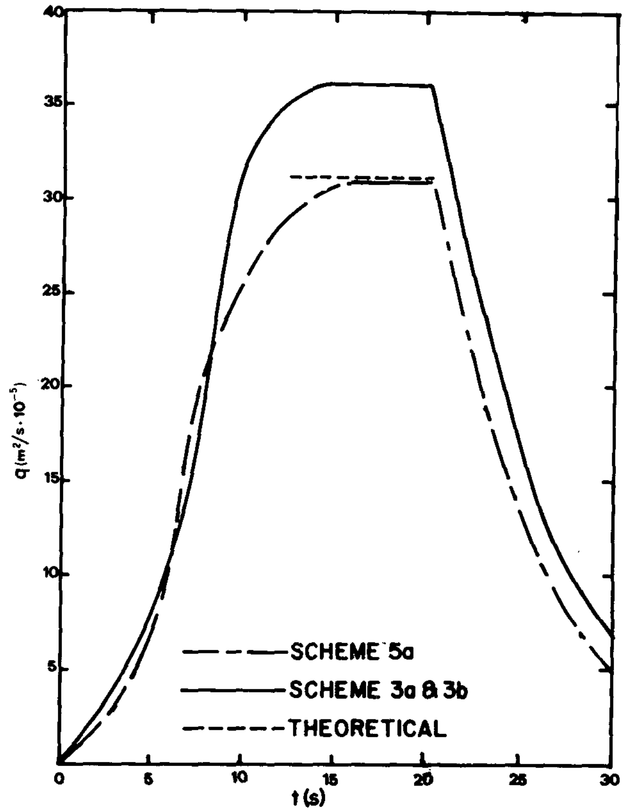

Figure 7
Problem 2: Discharge variation at the outlet

results. Henceforth only schemes 3b and 5a are used. Both schemes yielded satisfactory results.

Problem 3

Uneven slope summit

Consider a section as shown in Fig. 8 with properties as shown. Let excess rainfall be as defined for problem 2.

The relative elevations of points were fed into the programs as input.

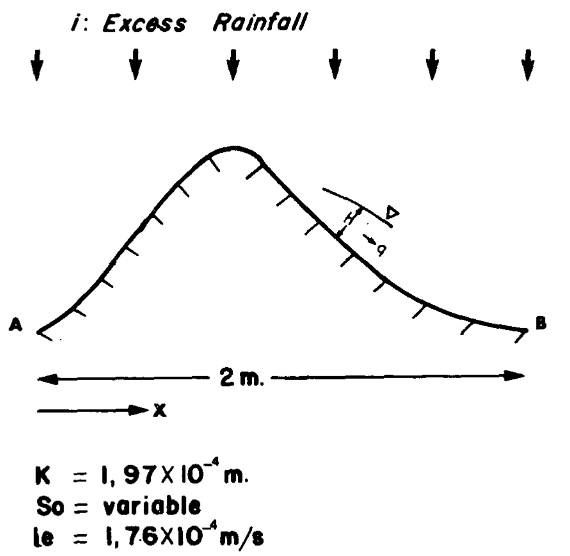

Figure 8
Problem 3: Uneven cross-section

Theoretical results

The uneven topography is approximated to two planes as shown in Fig. 9. The results are worked out as in problem one and are shown in Figures 10 and 11 together with the numerical results.

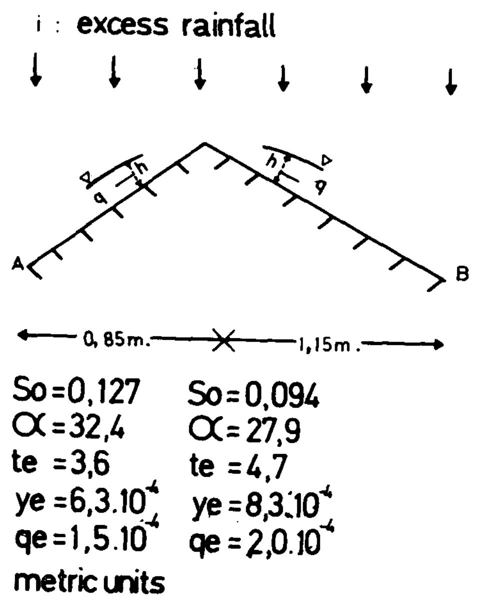

Figure 9
Problem 3: Approximation of cross-section by planes

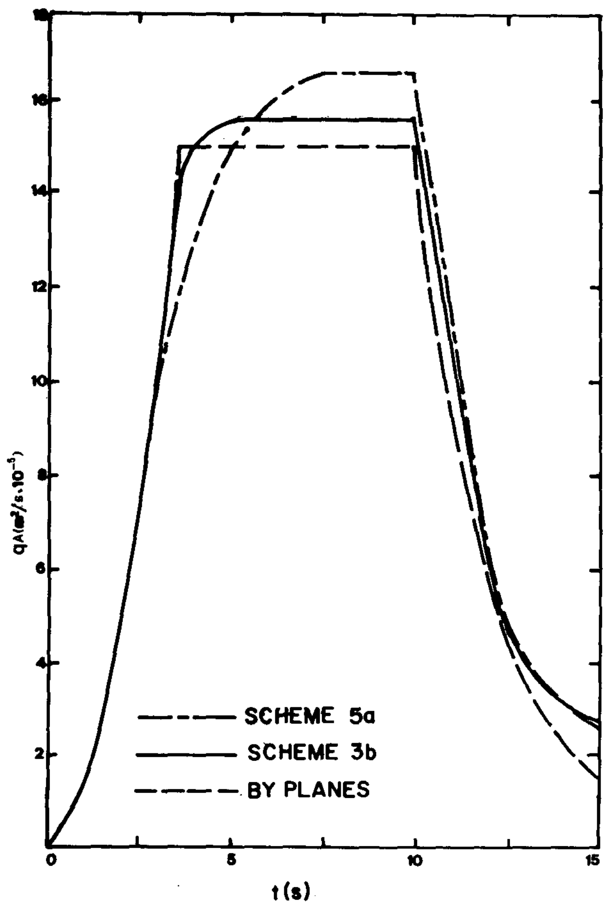

Figure 10
Problem 3: Discharge variation at point A

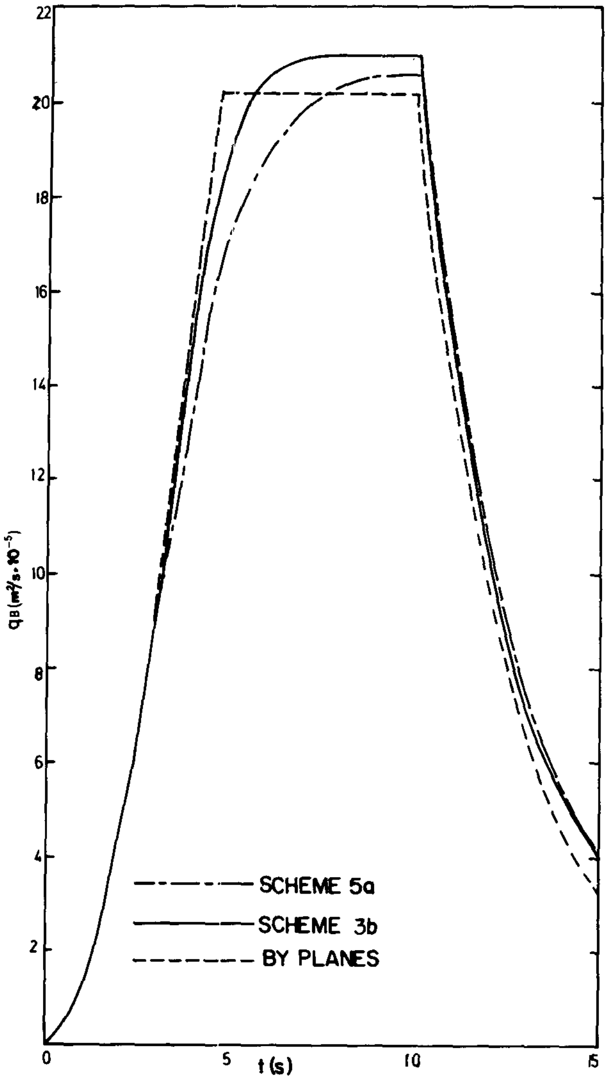

Figure 11
Problem 3: Discharge variation at point B

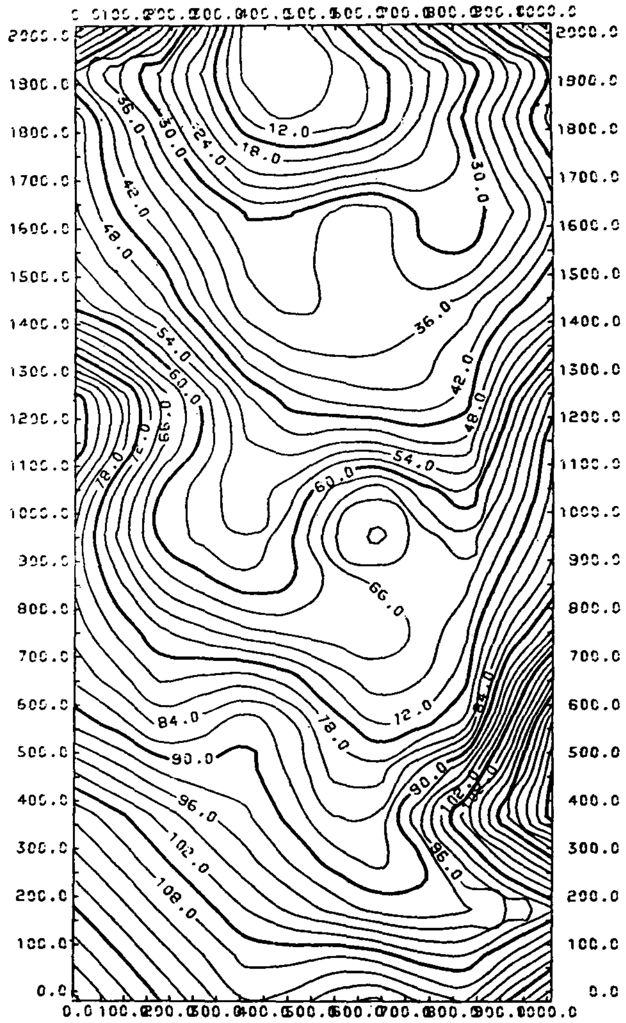

Figure 12
Contour plot of topography

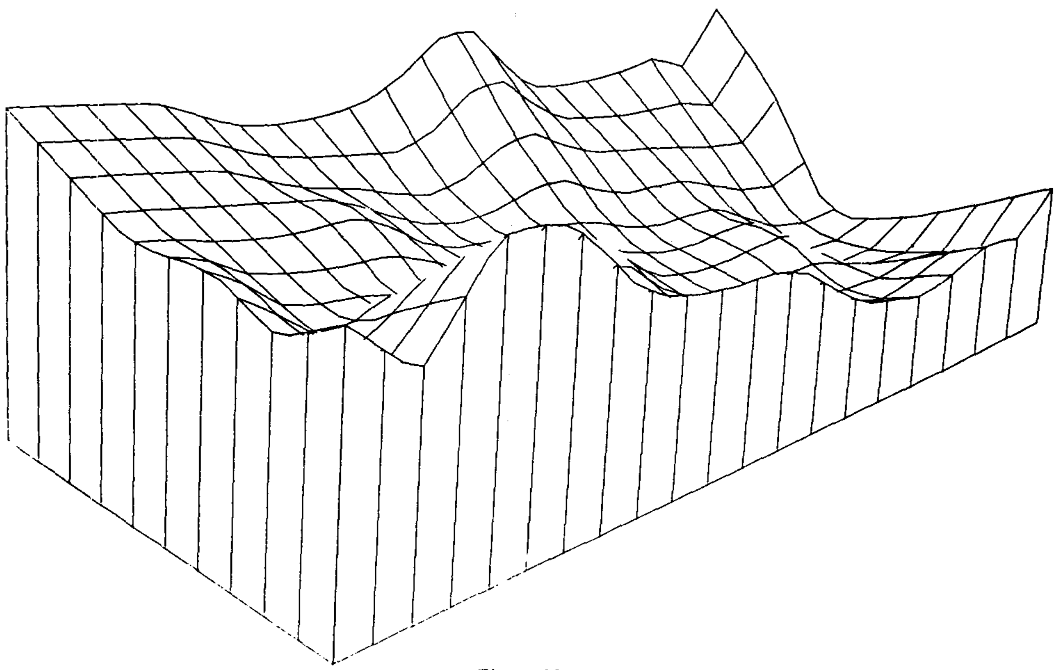

Figure 13
Topography in 3-Dimensions

Both methods seem quite accurate. They will be used in the next section in the development of the 2-dimensional scheme. Ideas employed in the compiling of the computer programs will also be used.

2-Dimensional model

The basic ideas developed in the previous sections together with equations (12), (17), (18) and (19) were used to create a 2 dimensional kinematic flow routing model for overland flow. A Newton-Raphson iterative technique was used in estimating discharges. Typical outputs for a square 2x1 km catchment are shown in Figs. 12 to 17. The program is flexible and can consider spatial and time variation of all input parameters.

The difference scheme used is a development of the one-dimensional 3b Backward Central explicit finite difference scheme explained in previous sections. The use of a scheme based on scheme 5a Diffusive scheme yielded unstable results.

Verification

The mathematical model was verified by using available data from experimental catchments (Constantinides and Stephenson, 1981). The results for one of these catchments, the South Parking Lot No. 1 (SPL1) at John Hopkins University documented by Schaake (1965) is described here.

The catchment layout and storm data were obtained from Harley *et al.*, 1970, Grace *et al.*, 1966, and Schaake (1965). The catchment layout and the results from three different storms are shown in Figures 18 to 22.

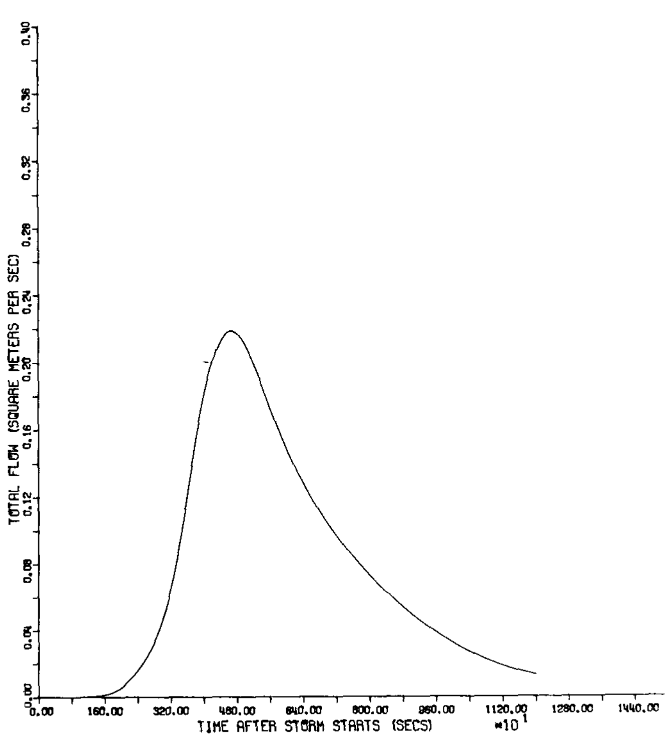

Figure 15
Runoff hydrograph at the outlet

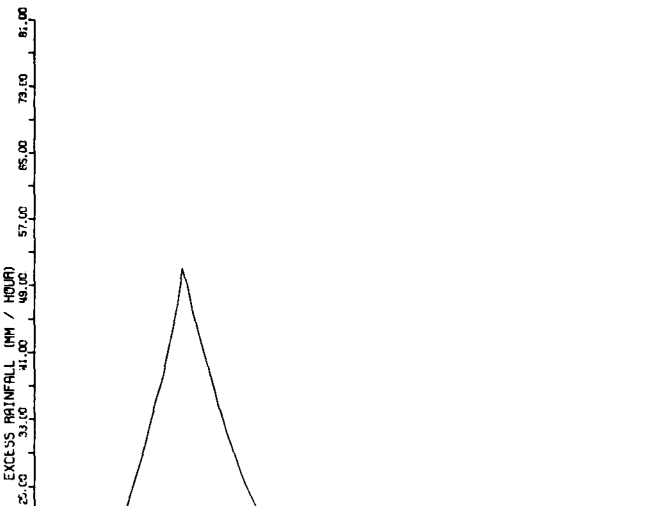

Figure 14
Excess rainfall variation with time over the catchment

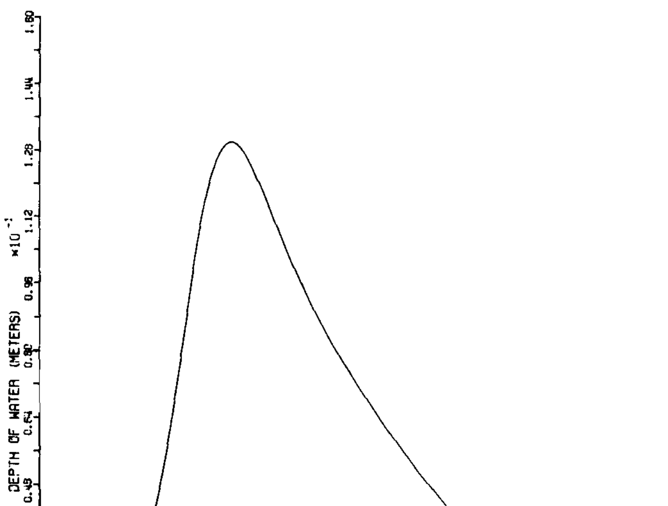

Figure 16
Water depth variation at the outlet

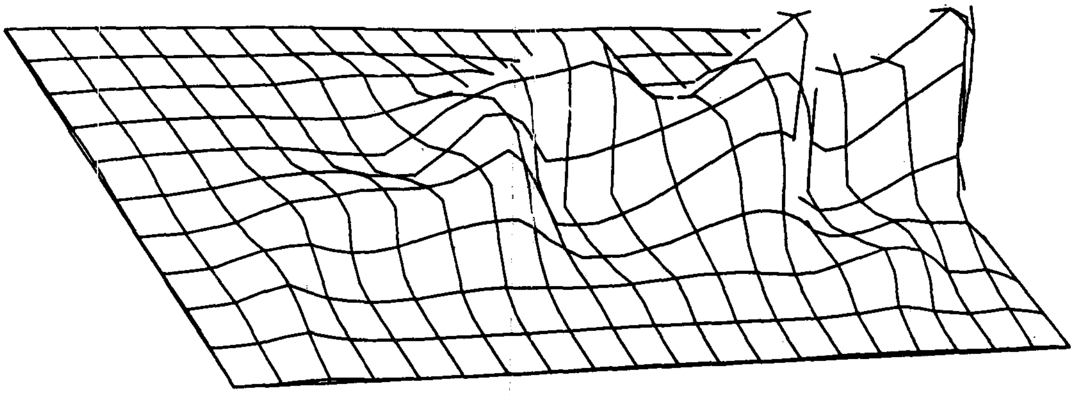

Figure 17
Water depth variation at t = 8 min over the catchment

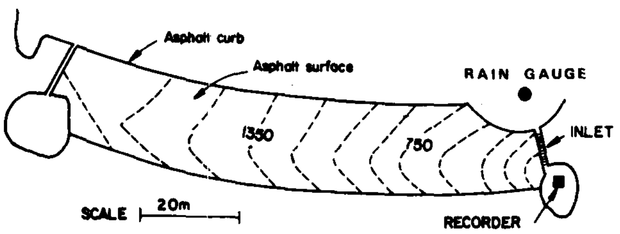

Figure 18
Plan view of the drainage area

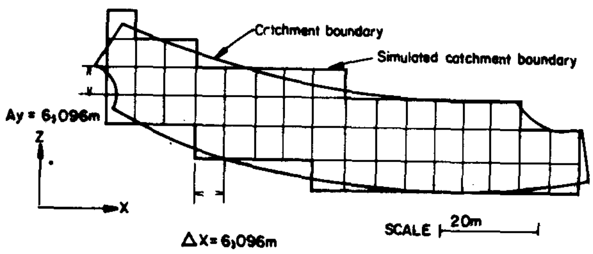

Figure 19
Grid used superimposed on catchment

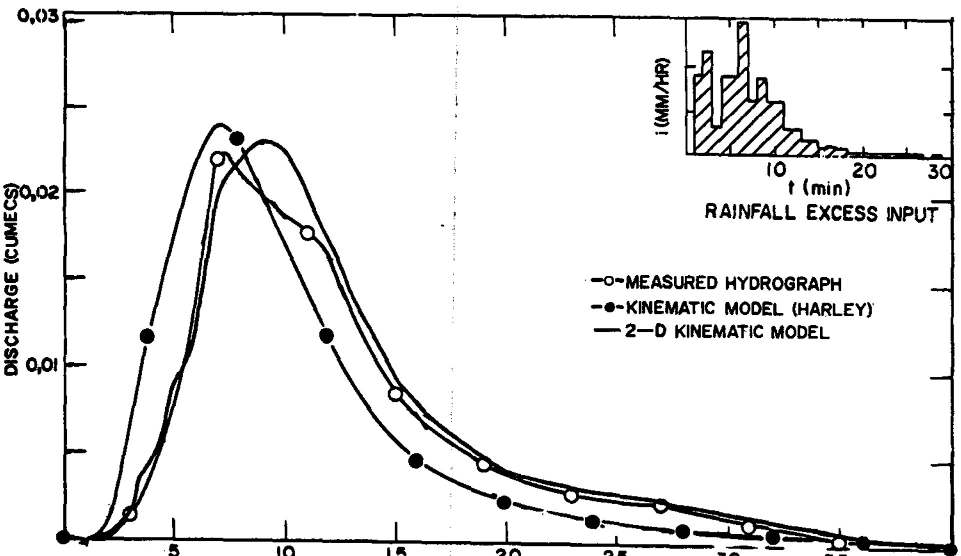

Figure 20
Measured and computed discharge hydrographs from SPL1 for storm No. 7

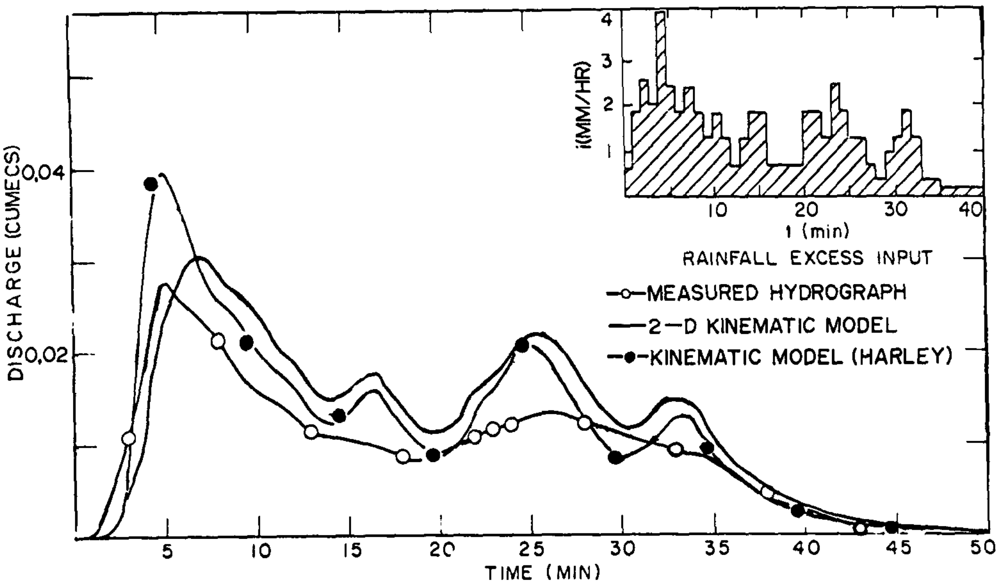

Figure 21
Measured and computed discharge hydrographs from SPL1 for storm No. 6

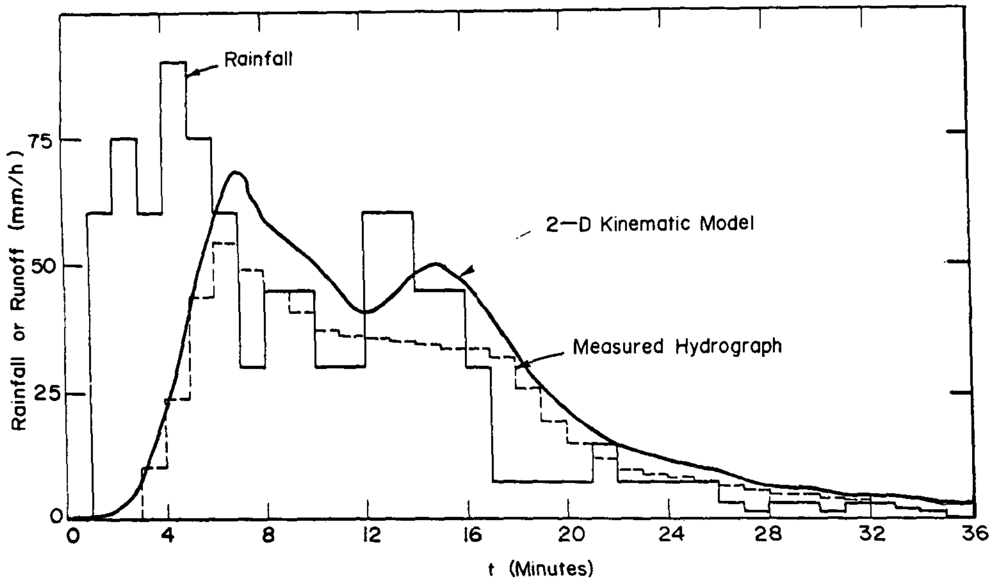

Figure 22
Measured and computed discharge hydrographs from SPL1 for storm No. 9

Extension and scope of model

The model results were also compared with results of other experimental catchments. The model was also compared with solutions of existing two-dimensional runoff models obtained by analytical solutions or by combinations of a number of one-dimensional flows.

The input to the model i.e. storm patterns, losses and roughness is being studied so that the model can be used in natural catchments. The model is also being adapted to account for canalization, deviations and discontinuities so that it can be used in urban catchments as well.

Conclusions

A model was developed for routing excess rainfall off two-dimensional planes using Kinematic equations. Numerical solutions to the equations can be performed simply and economically using a backward-central finite difference scheme.

The model is suitable for natural catchments of various shapes where the catchment characteristics such as elevation, roughness and infiltration vary continuously. Temporal and space variation of storms is also permitted, enabling the hydrologist to select the worst combination of storm movements for design.

The model is superior to empirical methods such as unit hydrographs and black box methods as it accounts for input variation usually assumed to be constant.

Coupled to a three dimensional plotting program the model provides a suitable method of visualizing the rainfall-runoff process off varying topography. The model can readily be extended to allow for channels and 'pans'. It is anticipated that such models will provide a better understanding of the hydrological runoff process.

Acknowledgements

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