An approach to the mathematical expression of recession curves

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Abstract

A recession curve of any origin can be mathematically expressed by a superimposition of the exponential equation $Q = Q_0 e^{-\alpha t}$, and evaluated from its basic parameters α – depletion coefficient and Q_0 – initial discharge. The expression of curves of various gradients and curvature by the same formula enables their comparison on an equal basis. In order to expand this facility the method for the construction of the mean and envelope recession curves from curves expressed by the above-mentioned relationship has been developed.

Introduction

The recession limb of a river discharge hydrograph comprises all the components of the total runoff. Base flow is represented by the lower portion of the recession curve designated as the depletion curve. Both terms, "recession curve" and "depletion curve", have been used in this paper accordingly. The general term, recession curve, has been used while describing the construction of the curve.

The terms "value of α" and "variability of α" used in the subsequent text should be interpreted in the following way. The term value of α refers to α of a single recession curve or more frequently to α of a mean recession curve presented on a semi-logarithmic scale as a straight line or a curve. In the case of a curve the value of α refers to individual $\alpha_1, \alpha_2, \ldots, \alpha_n$ or averages $\bar{\alpha}_1, \bar{\alpha}_2 \ldots, \bar{\alpha}_n$ (not to the average

$$\bar{\alpha}_i = \frac{1}{n}\sum_{i=1}^{n} \alpha_i).$$

The term variability of α refers to a dispersion of values of α or $\alpha_1, \alpha_2, \ldots, \alpha_n$ of individual recession curves from the value of $\bar{\alpha}$ or $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ of the mean recession curve of the particular set of curves. The term variability of α does not refer to a change in the value of α within a single recession curve resulting in $\alpha_1, \alpha_2, \ldots, \alpha_n$.

The equation:

$$Q = Q_0 e^{-\alpha(t - t_0)} \quad \text{(1)}$$

where

Q_0 = initial discharge at start of period at time t_0
Q = discharge at end of period at time t
e = base of natural logarithm
α = depletion coefficient

has been derived for the depletion of ground-water storage for unsteady laminar flow under:

- confined situations;
- unconfined situations where the ground-water level undergoes a small change during depletion.

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If time $t_0 = 0$ then

$$Q = Q_0 e^{-\alpha t} \quad \text{(2)}$$

The structure of the formula for the expression of α in Eq. (1) points to some factors controlling the course of depletion. The coefficient α for the confined aquifer according to Schoeller (1962) has a form:

$$\alpha = \frac{K \sin\beta}{m\,l} \quad \text{(3)}$$

where

K = coefficient of permeability
m = storage coefficient
l = length of saturated aquifer
β = dip angle of the aquifer

Eq. (1) for the depletion from an unconfined aquifer was derived from the basic non-linear partial differential equation of unsteady filtration flow with an unconfined water level in an anisotropic rock environment by Boussinesq (1904).

After its simplification to the one-dimensional unsteady non-linear form for the isotropic aquifer with a horizontal impermeable bed and using an assumption that the ground-water level does not change significantly due to depletion, the one-dimensional equation was linearised. The obtained form was transformed by further development to the exponential Eq. (1) with

$$\alpha = 2{,}467 \frac{K\,h_0}{m\,X^2} \quad \text{(4)}$$

where h_0 and X represent the saturated thickness and length of the aquifer.

Some discrepancies remain, because of the incorporation of the Dupuit-Forchheimer assumptions (Chow, 1964) upon which the one-dimensional solution is based and due to linearisation of the basic non-linear equation governing seepage flow. In both Eq. (3) and Eq. (4) α is directly proportional to the coefficient of permeability and inversely proportional to the storage coefficient.

An extremely rough simplification of the depletion of an aquifer can be simulated by the depletion of a reservoir through a porous cork (Roche, 1963). Depletion is expressed by Eq. (1) with

$$\alpha = \frac{K\,P\,s}{S\,L} \quad \text{(5)}$$

where

P = porosity
s and L = cross sectional area and length of the porous cork
S = horizontal sectional area of the reservoir.

Construction of the recession curve using the principle of superposition

Eq. (2) can be expressed in the logarithmic form as follows:

$$\log Q = \log Q_0 - 0{,}4343\, t\, \alpha$$

The plot of this curve on a semi-logarithmic coordinate system is a straight line. The depletion coefficient α is the gradient of this line and the equation is valid for the whole curve. The second parameter of the recession curve i.e. initial discharge Q_0 represents an ordinate of this curve at time $t = 0$ (Fig. 1).

Q_0 = 59.6 l/s
Q_t = 51.1 l/s
t = 161 days
α = 0.000955721

$Q = 59.6\, e^{-0.000956\, t}$

LEGEND:
A – Empirical curve
E – Analytical curve

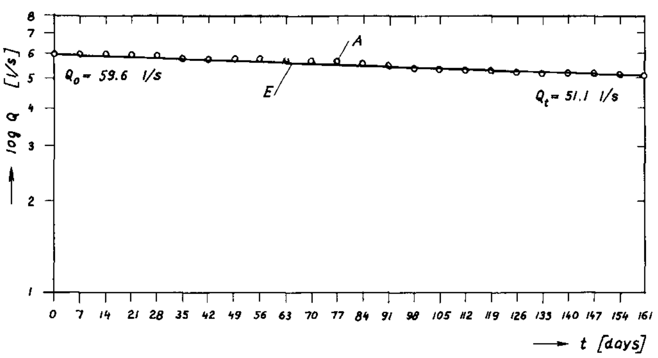

Figure 1
Depletion curve. Spring: Slavianska No. 3, Czechoslovakia, 27.6.1973–7.11.1973.

Generally a straight line represents the depletion from a homogeneous rock environment. The more permeable an aquifer the steeper the curve and the higher the value of α.

A heterogeneous aquifer is indicated by a curved line when the recession is plotted on a semi-logarithmic scale. With this situation the value of α is not constant for the whole curve but is a function of time.

Parameters Q_{0i} and α_i can be determined in the following way. Time on a linear scale and discharge on a logarithmic scale are plotted onto the chart (Fig. 2), (Tab. 1). The lower part of the empirical curve designated A is extended into the straight line B. The line B intersects the logarithmic axis at the point Q_{01}, and α_1 is the gradient of the line. For the remaining points on curve A, a curve C is obtained by differentiating the ordinates on curve A and straight line B. Straight line D fitted through the lower part, or the entire curve C, enables one to determine Q_{02} and α_2 the same way as in the previous case for the line B.

The value of α_i is computed from the following equation:

$$\alpha_i = \frac{\log Q_{0i} - \log Q_{ti}}{0{,}4343 t_i} \quad \ldots\ldots (6)$$

where Q_{ti} denotes discharges, on a lower part of the curves A and C, corresponding to the end of time interval t_i. The other quantities used in the procedure are as shown in Fig. 1 and Fig. 2.

In this manner, i.e. by the superimposition of Eq. (2), the entire curve A can be expressed in the form of a series of straight lines. Discharge Q at time t can be calculated from the equation of an analytical recession curve E obtained by the described procedure. The equation takes the form of

$$Q = Q_{01}e^{-\alpha_1 t} + Q_{02}e^{-\alpha_2 t} + \ldots + Q_{0n}e^{-\alpha_n t} \quad \ldots\ldots (7)$$

Q_{0i} and α_i ($i = 1, 2, \ldots, n$), correspond to the i^{th} line within the series of straight lines, on a semi-logarithmic scale, obtained by the superimposition of Eq. (2). The curve starts from the value $Q_0 = \sum_{i=1}^{n} Q_{0i}$ corresponding to the time $t = 0$ which is the beginning of the depletion.

A transition from direct runoff to base flow is usually accompanied by a change in curvature of the recession curve. A similar effect is observed on depletion curves due to the heterogeneity of an aquifer. Because the change in the value of α can be identified, a more accurate adjustment of Q_0 during the construction of the depletion curve can be made.

Q_0 = 2.02 l/s
Q_{01} = 1.32 ls
Q_{t1} = 0.79 l/s
t = 119 days
α_1 = 0.00434

Q_{02} = 0.70 l/s
Q_{t2} = 0.108 l/s
t = 28 days
α_2 = 0.06654

$Q = 0.70\, e^{-0.06654\, t} + 1.32\, e^{-0.00434\, t}$

LEGEND:
A = Empirical curve
B – Curve for α_1
C – Curve of differences
D – Curve for α_2
E – Analytical curve

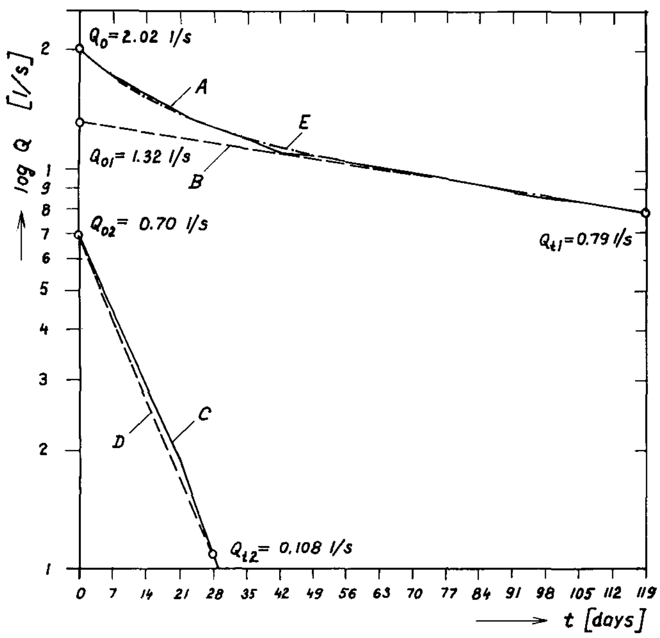

Figure 2
Depletion curve. Spring: Malý Roštun, Czechoslovakia, 21.8.1957–18.12.1957

TABLE 1
DEPLETION CURVE. SPRING: MALÝ ROŠTUN, CZECHOSLOVAKIA, 21.8.1957–18.12.1957 (INDEX TO FIG. 2)

t (days)	A (ℓ/s)	B (ℓ/s)	C (A–B) (ℓ/s)	D (ℓ/s)	E (B + D) (ℓ/s)	Deviation (A–E) (ℓ/s)
0	2,02	1,324	0,696	0,696	2,020	0,00
7	1,73	1,284	0,446	0,439	1,723	0,01
14	1,54	1,246	0,294	0,276	1,522	0,02
21	1,40	1,208	0,192	0,173	1,381	0,02
28	1,28	1,172	0,108	0,109	1,281	0,00
35	1,21	1,137	0,073	0,068	1,205	0,00
42	1,11	1,103	0,007	0,043	1,146	−0,04
49	1,08	1,067	0,013	0,027	1,094	−0,01
56	1,04	1,035	0,005	0,017	1,057	−0,02
63	1,01	1,004	0,006	0,011	1,015	0,00
70	0,98	0,974	0,006	0,007	0,981	0,00
77	0,95	0,045	0,005	0,004	0,949	0,00
84	0,92	0,917	0,003	0,003	0,920	0,00
91	0,88	0,889	−0,009	0,002	0,891	−0,01
98	0,85	0,863	−0,013	0,001	0,864	−0,01
105	0,83	0,837	−0,007	0,000	0,837	−0,01
112	0,81	0,812	0,002	0,000	0,812	0,00
119	0,79	0,788	0,002	0,000	0,788	0,00

If Q_0 and α are known the volume of ground water available in an aquifer at an instant $t = 0$, corresponding to Q_0, can be evaluated according to the following equation:

$$V = \sum_{i=1}^{n} [Q_{0i} \int_0^{\infty} e^{-\alpha_i t}\, dt] = \sum_{i=1}^{n} \frac{Q_{0i}}{\alpha_i k} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where k is a constant for the time unit. The dimension of α is a reciprocal value of time [1/t].

Analyses of depletion curves show that in general, for a complete expression of the depletion curve, two or three straight lines adequately represent the curve. Accordingly, two or three values of α can characterise depletion patterns.

Construction of mean and envelope recession curves

The solution is based on analytical recession curves obtained by the superimposition of Eq. (2). Curves are graphically presented in the semi-logarithmic coordinate system (Fig. 3), (Tab 2) and mathematically can be expressed according to the following equation:

$$Q_j = Q_{01j}e^{-\alpha_{1j}t} + Q_{02j}e^{-\alpha_{2j}t} + \ldots + Q_{0nj}e^{-\alpha_{nj}t} \quad \ldots. (9)$$

where (j = I, II, ..., N) denote the value of j^{th} curve.

The number of curves depends on the occurrence of uninfluenced periods during the observation. Precipitation during the depletion often deforms the curve considerably and its influence decreases more or less gradually. The number of terms within the equation depends on the curvature of the curve. It should be noted that individual curves are designated by Roman numerals I, II, ..., N.

Each curve expressed by Eq. (9) starts from some value $Q_{0j} = Q_{01j} + Q_{02j} + \ldots + Q_{0nj}$ corresponding to the time $t = 0$. A negligible error may be incurred by the extrapolation of individual curves, according to α, up to the value Q_{0max} i.e. the

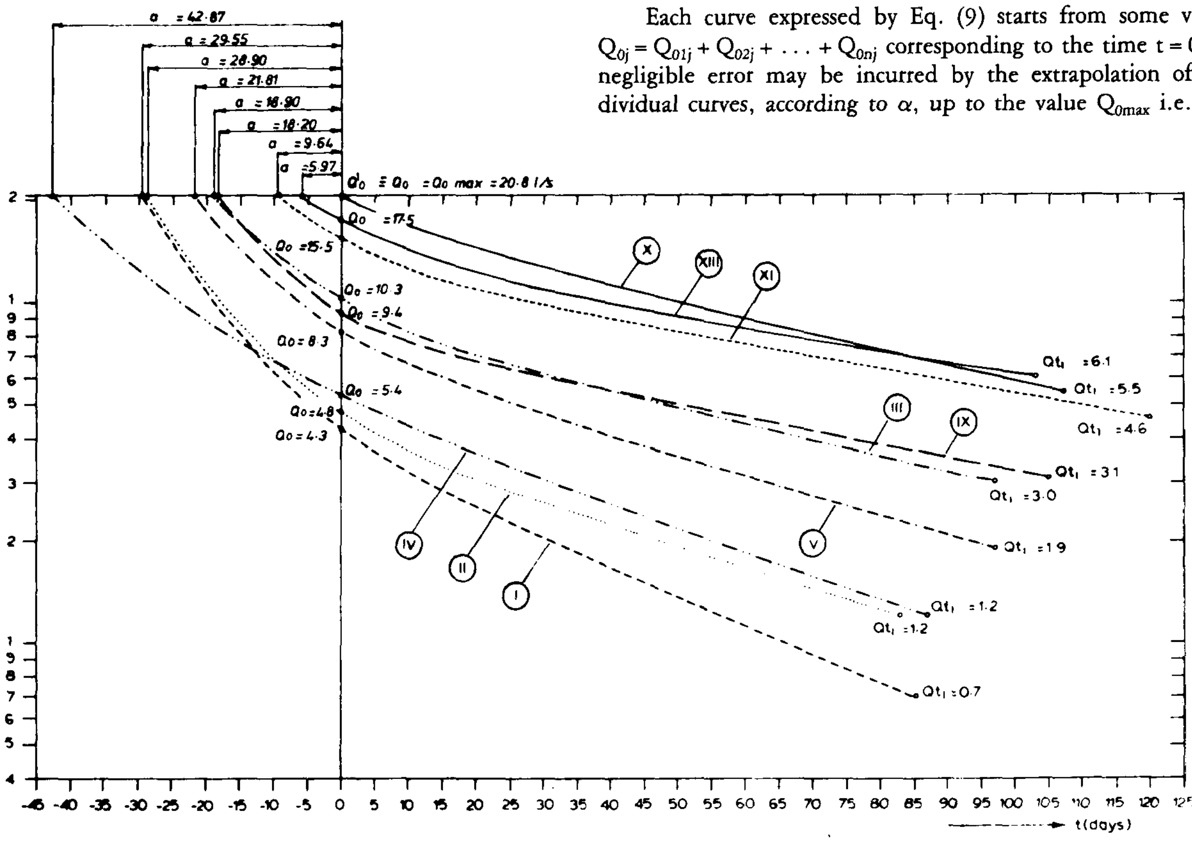

Figure 3
Extrapolation of depletion curves. Catchment A at the Luano Catchments, Zambia, 1967–1979.

highest initial discharge Q_{0j} within the set of curves (Fig. 3).

The symbol a_j denotes the time interval between the new initial discharge $Q'_{0j} = Q_{0max}$, obtained by extrapolation, and the origin of the coordinate system. The value a_j is determined through an iteration procedure, using Eq. (9) for each curve separately, by substituting the value Q_{0max} for Q_j, and the negative values a_j for time t. The general form of the equation for the calculation of a_j is

$$Q'_{0j} = Q_{01j}e^{\alpha_{1j}a_j} + Q_{02j}e^{\alpha_{2j}a_j} + \ldots + Q_{0nj}e^{\alpha_{nj}a_j} \quad \ldots\ldots (10)$$

After values a_j are known, the values Q'_{0ij}, corresponding to time $t = -a_j$, are computed for each extrapolated curve from individual terms of Eq. (10) i.e. using the relation

$$Q'_{0ij} = Q_{0ij}e^{\alpha_{ij}a_j}$$

By the substitution of Q_{0ij} with Q'_{0ij} and t with t' (where $t' = t + |a_j|$) in Eq. (9), the curves are shifted by the time intervals a_j. Their initial discharges Q'_{0j} obtained by the extrapolation now refer to the time t = 0 at the origin of the coordinate system (Fig. 4). The equation of extrapolated and shifted curves has the form

$$Q_j = Q'_{01j}e^{-\alpha_{1j}t} + Q'_{02j}e^{-\alpha_{2j}t} + \ldots + Q'_{0nj}e^{-\alpha_{nj}t} \quad \ldots (11)$$

Parameters of curves affected by the extrapolation are designated by an apostrophe ('). The described procedure brings the set of curves to a position suitable for the construction of the mean and envelope recession curves.

The equation of the mean recession curve is obtained by averaging the basic parameters Q_{0ij} and α_{ij} within the set of curves expressed by Eq. (11). Its form is as follows:

$$\bar{Q} = \bar{Q}'_{01}e^{-\bar{\alpha}_1 t} + \bar{Q}'_{02}e^{-\bar{\alpha}_2 t} + \ldots + \bar{Q}'_{0n}e^{-\bar{\alpha}_n t} \quad \ldots\ldots (12)$$

where

$$\bar{Q}'_{01} = \frac{1}{N}\sum_{j=1}^{N} Q'_{01j}\,;\ \bar{Q}'_{02} = \frac{1}{N}\sum_{j=1}^{N} Q'_{02j}\,;\ \ldots\,;\ \bar{Q}'_{0n} = \frac{1}{N}\sum_{j=1}^{N} Q'_{0nj}\,;$$

and

$$\bar{\alpha}_1 = \frac{1}{N}\sum_{j=1}^{N} \alpha_{1j}\,;\ \bar{\alpha}_2 = \frac{1}{N}\sum_{j=1}^{N} \alpha_{2j}\,;\ \ldots\,;\ \bar{\alpha}_n = \frac{1}{N}\sum_{j=1}^{N} \alpha_{nj}$$

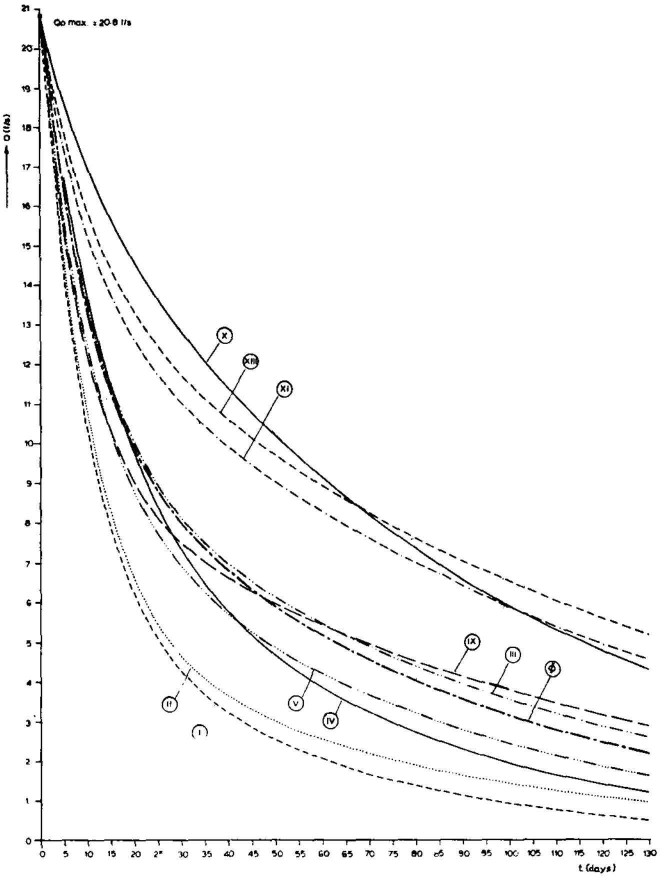

Figure 4
Mean depletion curve. Catchment A at the Luano Catchments, Zambia, 1967–1979.

Envelope recession curves are usually identical with the upper and lower curve within the set of curves initiating from Q_{0max}. In the case where the upper or lower border of the range of depletion is created by more curves crossing one another (Fig. 5) the envelope curve is constructed from points on the boundary of crossing curves by the method of superimposition of Eq. (2).

The construction of the mean and envelope recession curves by the above described method (Petras, 1976) is demonstrated on nine curves from the outlet section of the representative Luano

TABLE 2
BASIC PARAMETERS OF DEPLETION CURVES. CATCHMENT A AT THE LUANO CATCHMENTS, ZAMBIA, 1967–1979 (ALSO INDEX TO FIG. 3, FIG. 4 AND FIG. 5)

Depletion curve	Period	Q_0 (ℓ/s)	Q_1 (ℓ/s)	Q_{01} (ℓ/s)	t_1 (days)	Q_{02} (ℓ/s)	1	2	a (days)	Q'_{01} (ℓ/s)	Q'_{02} (ℓ/s)	t'_1 (days)
I	11.5– 4.8.1967	4,30	0,70	3,60	85	0,70	0,01927	0,10242	29,55	6,36	14,44	114,55
II	3.5–25.7.1968	4,80	1,20	3,90	83	0,90	0,01420	0,09718	28,90	5,88	14,92	111,90
III	10.5–15.8.1969	10,30	3,00	8,40	97	1,90	0,01061	0,09065	18,90	10,26	10,54	115,90
IV	30.4–26.7.1970	5,40	1,20	4,90	87	0,50	0,01617	0,07210	42,87	9,80	11,00	129,87
V	25.4–31.7.1971	8,30	1,90	7,00	97	1,30	0,01344	0,09963	21,81	9,38	11,42	118,81
IX	10.5–23.8.1975	9,40	3,10	7,90	105	1,50	0,00891	0,11198	18,20	9,29	11,51	123,20
X	21.5– 5.9.1976	20,80	5,50	17,50	107	3,30	0,01082	0,09605	0,00	17,50	3,30	107,00
XI	24.4–22.8.1977	15,50	4,60	12,60	120	2,90	0,00840	0,09348	9,64	13,66	7,14	129,64
XIII	1.5–12.8.1979	17,50	6,10	13,50	103	4,00	0,00771	0,08557	5,97	14,13	6,67	108,97
Φ	(mean curve)	20,80					0,01217	0,09434	—	10,70	10,10	—
VI	26.5–22.8.1972	3,50	0,90	3,50	88	—	0,01543	—	—	—	—	—
VII	5.4–31.5.1973	2,70	0,30	2,70	56	—	0,03932	—	—	—	—	—
VIII	12.6– 6.8.1974	0.90	0,34	0,90	55	—	0,01770	—	—	—	—	—
XII	22.5–14.8.1978	15,00	7,80	15,00	84	—	0,00778	—	—	—	—	—

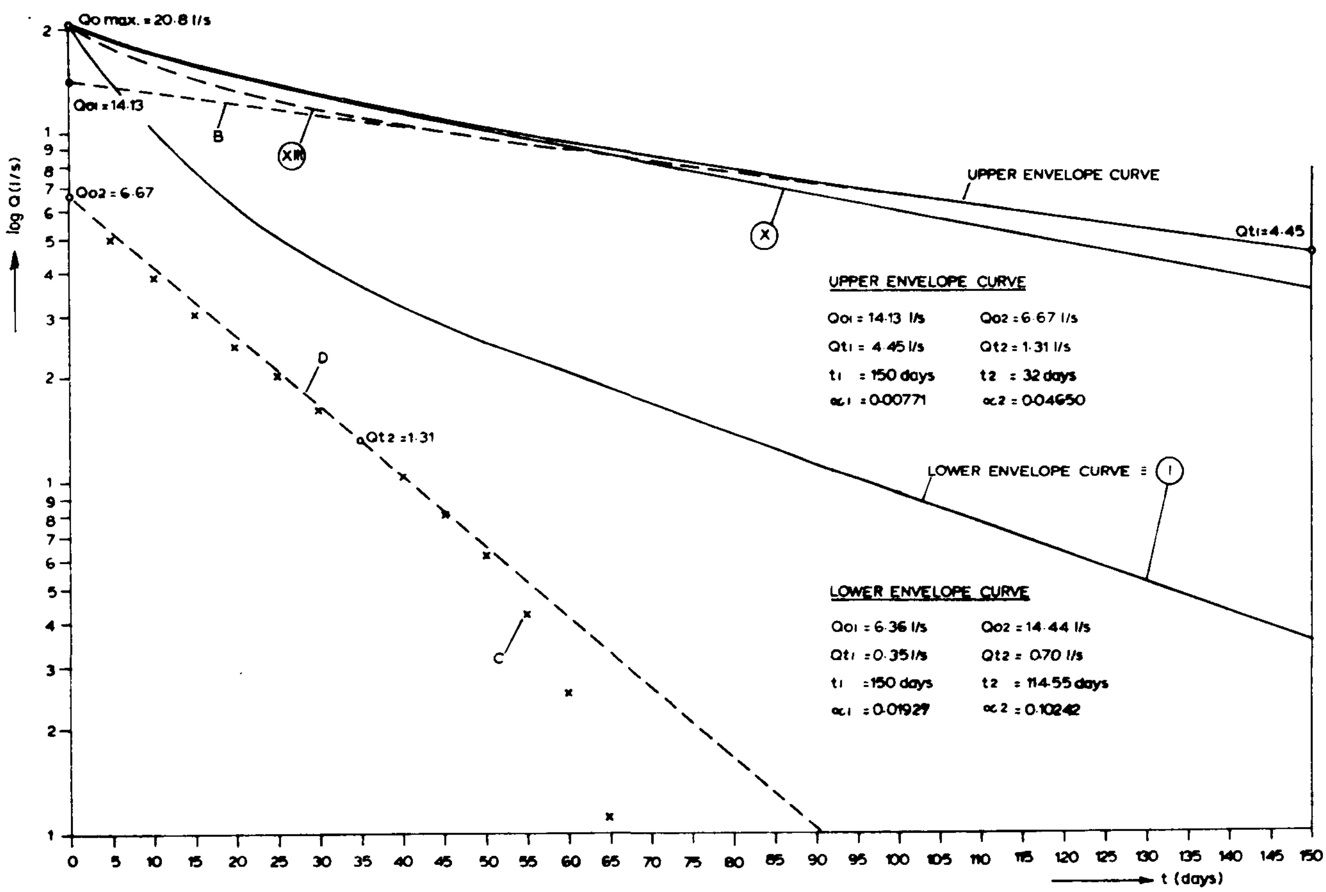

Figure 5
Construction of envelope depletion curves. Catchment A at the Luano Catchments, Zambia, 1967–1979.

Catchment A in Zambia (Figs. 3, 4, 5, Table 2), (Petras, 1981). One superimposition of Eq. (2) was adequate to represent the depletion in this case.

The mean recession curve can also be constructed in a simple way by averaging the basic parameters of evaluated curves (according to individual terms of their equations) without extrapolation of individual curves. In order to obtain the initial value of the discharge of the mean recession curve identical to the value Q_{0max}, only the extrapolation of the mean recession curve up to the value Q_{0max} is required and initial discharges Q_{0i} for this point are computed. However, the set of recession curves adjusted in the rectangular coordinate system to a position when curves initiate from the same point Q_{0max} allows one to construct the envelope recession curves and graphically express the range of recession.

Conclusion

The approach to the mathematical expression of recession curves discussed in this paper was used and factors affecting the value of α and its variability were studied through an analysis of 333 depletion curves from sixty-nine springs draining geologically and tectonically different rock environments in Slovakia (Kullman and Petras, 1979). Limestones and dolomites were represented by fifty-six springs, sandstones and conglomerates by eight and andezites by four springs. Each spring and adjacent region were evaluated from a hydrological and geological point of view including an evaluation of tectonic conditions. Further results were obtained through the analysis of 52 depletion curves from four Representative Luano Catchments in Zambia (Petras, 1981).

The analysis shows that the close relationship between the depletion coefficient and the geological character of the rock environment can be established. The value of α depends mostly on the character, degree and extent of disintegration of the rock environment of an aquifer and on its lithological, tectonic and geometric characteristics. The influence of a soil cover is more or less significant depending on its thickness and properties. Variability of α within the framework of the aquifer during different periods of depletion is mostly due to the mutual influence of irregularities in the geological character of the rock environment, aquifer saturation coefficient, the surface of the infiltration area and meteorological phenomena. The results indicate that depletion coefficient's value and variability can be considered as a characteristic of the aquifer.

The analysis also indicates broader utilization of the described procedure for the solution of hydrological and hydrogeological problems as for example the evaluation of a ratio of ground-water depletion from large fissures to the ground water from small fissures in a fractured aquifer, hydrograph analyses and low discharge forecasts.

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