Parameter estimation for the general extreme value distribution

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Abstract

This study is concerned with the estimation of the parameters of the general (or generalised) extreme value (GEV) distribution by the methods of maximum likelihood (ML) and probability-weighted moments (PWM) for complete and type I censored samples.

For complete samples, the PWM provided estimators which are less biased than the ML estimators. For the variances/covariances of the parameter estimators, the PWM had a comparable efficiency to the ML. However, for the extreme quantiles, the PWM estimators had much larger variances.

For censored samples, the extension provided in this study for the PWM did not perform satisfactorily in terms of the bias and variances of the estimators. The ML, on the other hand, still functioned well. It can reduce the bias and even the variances of the estimators at some censoring levels.

Finally, the Akaike information criterion, used along with the ML estimators, can identify the extremal models with a high accuracy level for both complete and censored samples.

Introduction

The magnitude of extreme events may be taken, to a reasonable approximation, as being distributed according to one of the three extreme-value distributions defined by Fisher and Tippett (1928). The three extremal models are as follows:

- The extreme value type I (EV1) distribution is widely known as the Gumbel distribution after Gumbel (1958) first applied it to flood frequency analysis. This two-parameter distribution is advantageous from a theoretical point of view since it is simple in the sense that it has both the density and distribution function in closed form; moreover, it has all useful moments also in closed form. Its application is, however, limited because it has only two parameters and a constant skewness coefficient and thus cannot be flexible enough to represent adequately a number of extreme sequences, such as flood data.
- The extreme value type II (EV2) distribution is sometimes known as the Frechet distribution. It is applied to the estimation of maximum rainfall amount, since to a considerable extent rainfall amounts are "uncontrollable", and extremely high values may be recorded.
- The extreme value type III (EV3) distribution is the most frequently found in nature. It is also known as the Weibull distribution since Weibull (1951) introduced it to describe the behaviour of the breaking strength of materials.

Jenkinson (1955) combined these three extreme-value distributions into one, i.e., the general (or generalised) extreme value (GEV) distribution, by using a transformation and reparameterisation of the three-parameter Weibull distribution. The GEV distribution was recommended by the National Environmental Research Council (NERC, 1975) for representing the distribution of annual maximum flow sequences in the United Kingdom and Ireland.

In practical applications, the three parameters of the GEV must be estimated from sample data. For complete samples, Jenkinson's (1969) method of sextiles, method of probability-weighted moments (PWM) and method of maximum likelihood (ML) can readily be used. An assessment of the performance of these methods was partly made by Hosking *et al.* (1985). In this study, an extension of the PWM was made for the case of censored samples. Monte Carlo experiments were then carried out to compare the performance of the ML and PWM methods on such samples, and to investigate the effect of censoring on the estimates of the T-year flood. It should be noted that type I (rather than type II) censoring was considered in this study because of the following reasons:

- Results related to type II censoring are already available (Prescott and Walden, 1983).
- Type I censoring is commonly encountered with measurement devices. It is very useful in flood frequency analysis when historic information is incorporated (Leese, 1973; Condie and Lee, 1982; Stedinger and Cohn, 1986).

A definition of the two types of censoring is given in the next section.

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General considerations

Definition

The generalised extreme-value (GEV) distribution with parameters $a>0$, b and c has the distribution function given by:

$$F(X) = P(X \leqslant x) = \begin{cases} \exp\{-[1-b(x-c)/a]^{1/b}\} & (b \neq 0) \\ \exp\{-\exp[-(x-c)/a]\} & (b=0) \end{cases} \quad (1)$$

with x bounded by $c + a/b$ from above if $b>0$ and from below if $b<0$. Here c and a are location and scale parameters, respectively, and the shape parameter, b, determines which extreme-value distribution is represented. Fisher-Tippett types I, II and III correspond to $b = 0$, $b<0$, and $b>0$, respectively.

Censoring

Formal definitions of type I or type II censoring can be found in Kendall and Stuart (1979), and Condie and Lee (1982) where illustrations are also given. Basically, a sample is censored when a known number of the smallest or largest observations are missing (David, 1970). If the censoring takes place above or below a known value, then type I censoring occurs. In this case the number of missing values is a random variable. Type II censoring occurs when a fixed number of observations are removed. Correspondingly, the censoring point is a random variable.

In the present context of double censoring, let us consider a random sample of size N, where m values on the left of the lower point x_L, k values as the right of the upper point x_U are missing, and $n = N - m - k$ values in the middle are observed. If x_L and x_U are fixed, then the sample is censored according to type I. In this case, m, n and k are random variables. However, if m and k (and hence n) are fixed, then the sample is censored according to type II. In this case, x_L and x_U are two random variables. As type I censoring is considered in the rest of the paper, x_U and x_L are fixed.

Method of probability-weighted moments

a) Uncensored (complete) samples

The probability-weighted moments, a generalisation of the usual moments, were introduced by Greenwood *et al.* (1979). For the GEV distribution, they are given (Hosking *et al.*, 1985) by:

$$T_r = (r+1)^{-1} \{c + a[1 - (r+1)^{-b}\Gamma(1+b)]/b\}\ , \ b > -1$$

When $b \leqslant -1$, T_0 (the mean of the distribution) and the rest of the T_r do not exist.

Given a random sample of size N from the distribution, an estimator of T_r is most conveniently based on the ordered samples $x_1 \leqslant x_2 \leqslant \ldots x_N$. The statistic

$$t_r = N^{-1} \sum_{i=1}^{N} \frac{(i-1)(i-2)\ldots(i-r)}{(N-1)(N-2)\ldots(N-r)} x_i \qquad (2)$$

is an unbiased estimator of T_r (Landwehr *et al.*, 1979). In GEV case, only three estimators of PWM are needed, which are as follows:

$$t_0 = N^{-1} \sum_{i=1}^{N} x_i \qquad (3)$$

$$t_1 = N^{-1}(N-1)^{-1} \sum_{i=1}^{N} (i-1)x_i \qquad (4)$$

$$t_2 = N^{-1}(N-1)^{-1}(N-2)^{-1} \sum_{i=1}^{N} (i-1)(i-2)x_i \qquad (5)$$

From these equations, the PWM estimator of b can be approximated (Hosking *et al.*, 1985) by:

$$\hat{b} = 7{,}8590d + 2{,}9554d^2 \qquad (6)$$

where

$$\hat{d} = [(2t_1 - t_0)/(3t_2 - t_0)] - (\ln 2/\ln 3)$$

Given the estimator of b, the scale and location parameters can be obtained successively as:

$$\hat{a} = [(2t_1 - t_0)\hat{b}]/[\Gamma(1+\hat{b})(1 - 2^{-b})] \qquad (7)$$

$$\hat{c} = t_0 + \hat{a}[\Gamma(1+\hat{b}) - 1]/\hat{b} \qquad (8)$$

b) Censored samples

For samples subjected to censoring the necessary adjustments for PWM estimators t_0, t_1 and t_2 are needed. Suppose that m smallest and k largest observations are censored, then the summation in the PWM estimator expression should include only the uncensored portion; that is from $(m+1)$ to N, or 1 to $(n-k)$ for left censoring and right censoring respectively. For doubly censored samples the range will be changed from $(m+1)$ to $(N-k)$. The corresponding equations can be written as follows:

$$t_0 = N^{-1} \sum_{i=m+1}^{N-k} x_i \qquad (9)$$

$$t_1 = N^{-1}(N-1)^{-1} \sum_{i=m+1}^{N-k} (i-1)x_i \qquad (10)$$

$$t_2 = N^{-1}(N-1)^{-1}(N-2)^{-1} \sum_{i=m+1}^{N-k} (i-1)(i-2)x_i \qquad (11)$$

For left censoring, $k = 0$; and for right censoring, $m = 0$.

Method of maximum likelihood

By differentiation, the density function of the GEV distribution is obtained from Eq. 1 as:

$$f(x) = \exp\{-\exp(-y) + (-y)\}/[a(1-t)] \qquad (12)$$

where y, the reduced variable, is given by:

$$y = -b^{-1}\ln(1-t), \text{ and } t = bz,\ z = (x-c)/a \qquad (13)$$

The likelihood function of a random sample of size N, of which m lowest and k largest observations are censored, is proportional to (Kendall and Stuart, 1979):

$$\left[\int_{-\infty}^{x_L} f(x)dx\right]^m * \prod_{i=1}^{n} f(x_i) * \left[\int_{x_U}^{\infty} f(x)dx\right]^k$$

where x_L and x_U denote the left and right censoring points respectively and n is the number of uncensored observations:

$$n = N - m - k$$

It should be noted that for type I censoring, m, n and k are random variables while x_L and x_U are fixed.

The log likelihood function (to additive constants) is:

$$L = m\ln F(x_L) + \sum_{i=1}^{n} \ln f(x_i) + k\ln[1 - F(x_U)] \qquad (14)$$

The maximum likelihood equations are obtained by equating to zero the partial derivatives of L with respect to a, b and c. These are given by:

$$\begin{aligned} \partial L/\partial a &= mP_L + \Sigma P + kP_U = 0 \\ \partial L/\partial b &= mQ_L + \Sigma Q + kQ_U = 0 \\ \partial L/\partial c &= mR_L + \Sigma R + kR_U = 0 \end{aligned} \qquad (15)$$

In these expressions, the summation is taken over n uncensored observations. The different terms which were used in Eq. 15 are as follows:

$$\begin{aligned} P &= \partial \ln f(x)/\partial a = -1/a + (-1 + b + e^{-Y}) * y_a \\ Q &= \partial \ln f(x)/\partial b = y + (-1 + b + e^{-Y}) * y_b \\ R &= \partial \ln f(x)/\partial c = (-1 + b + e^{-Y}) * y_c \end{aligned} \qquad (16)$$

$$P_L = \partial \ln F(x_L)/\partial a = e^{-Y_L} * (y_L)_a$$
$$Q_L = \partial \ln F(x_L)/\partial b = e^{-Y_L} * (y_L)_b \quad (17)$$
$$R_L = \partial \ln F(x_L)/\partial c = e^{-Y_L} * (y_L)_c$$

$$P_U = \partial \ln[1 - F(x_U)]/\partial a$$
$$= -F(x_U) * e^{-Y_U} * (y_U)_a / [1 - F(x_U)]$$
$$Q_U = \partial \ln[1 - F(x_U)]/\partial b$$
$$= -F(x_U) * e^{-Y_U} * (y_U)_b / [1 - F(x_U)] \quad (18)$$
$$R_U = \partial \ln[1 - F(x_U)]/\partial c$$
$$= -F(x_U) * e^{-Y_U} * (y_U)_c / [1 - F(x_U)]$$

and

$$y_a = \partial y/\partial a = -z/[a(1-t)]$$
$$y_b = \partial y/\partial b = \{z/[b(1-t)]\} - y/b \quad (19)$$
$$y_c = \partial t/\partial c = -1/[a(1-t)]$$

For complete samples, Eqs. 17 and 18 are zeros; for censored samples, Eqs. 17 and 18 give the adjustments necessary to allow for left, right or double censoring. Left and right censorings are indicated by terms with subscripts L and U, respectively.

The ML equations can be solved using the algorithm developed by Hosking (1985) which modifies Newton's iterative procedure. The basic Newton-Raphson method solves the likelihood equations $\partial L/\partial \Theta = 0$ iteratively as follows:

$$\Theta_{j+1} = \Theta_j + d\Theta$$

where:

- $\Theta = [abc]^T$ is the parameter vector;
- $d\Theta = [-M]^{-1} \begin{bmatrix} \partial L/\partial a \\ \partial L/\partial b \\ \partial L/\partial c \end{bmatrix}_j$;
- j is the sequence number of the iteration; and
- M is the observed information matrix, i.e. the matrix of the second derivatives of L with respect to the parameters, evaluated at their current values.

$$\begin{aligned}
\partial^2 L/\partial a^2 &= m\partial P_L/\partial a + \Sigma\partial P/\partial a + k\partial P_U/\partial a \\
\partial^2 L/\partial a\partial b &= m\partial P_L/\partial b + \Sigma\partial P/\partial b + k\partial P_U/\partial b \\
\partial^2 L/\partial a\partial c &= m\partial R_L/\partial a + \Sigma\partial R/\partial a + k\partial R_U/\partial a \\
\partial^2 L/\partial b^2 &= m\partial Q_L/\partial b + \Sigma\partial Q/\partial b + k\partial Q_U/\partial b \\
\partial^2 L/\partial b\partial c &= m\partial R_L/\partial b + \Sigma\partial R/\partial b + k\partial R_U/\partial b \\
\partial^2 L/\partial c^2 &= m\partial R_L/\partial c + \Sigma\partial R/\partial c + k\partial R_U/\partial c
\end{aligned} \quad (20)$$

The derivatives based on the uncensored observations are given by:

$$\begin{aligned}
\partial P/\partial a &= (-1 + b + e^{-Y})y_{aa} - e^{-Y} * y_a * y_a + 1/a^2 \\
\partial P/\partial b &= (-1 + b + e^{-Y})y_{ab} - e^{-Y} * y_a * y_b + y_a \\
\partial Q/\partial b &= (-1 + b + e^{-Y})y_{bb} - e^{-Y} * y_b * y_b + 2y_b \\
\partial R/\partial a &= (-1 + b + e^{-Y})y_{ca} - e^{-Y} * y_c * y_a \\
\partial R/\partial b &= (-1 + b + e^{-Y})y_{cb} - e^{-Y} * y_c * y_c + y_c \\
\partial R/\partial c &= (-1 + b + e^{-Y})y_{cc} - e^{-Y} * y_c * y_c
\end{aligned} \quad (21)$$

where
$$\begin{aligned}
y_{aa} &= z(2-t)/[a^2(1-t)^2] \\
y_{ab} &= -z^2/[a(1-t)^2] \\
y_{bb} &= -z^2/[b(1-t)^2] - (2/b) * y_b \\
y_{ca} &= 1/[a^2(1-t)^2] \\
y_{cb} &= -z/[a(1-t)^2] \\
y_{cc} &= b/[a^2(1-t)^2]
\end{aligned}$$

The derivatives of P_L, Q_L and R_L corresponding to the left censoring are given by:

$$\begin{aligned}
\partial P_L/\partial a &= e^{-Y_L} * [(y_L)_{aa} - (y_L)_a * (y_L)_a] \\
\partial P_L/\partial b &= e^{-Y_L} * [(y_L)_{ab} - (y_L)_a * (y_L)_b] \\
\partial Q_L/\partial b &= e^{-Y_L} * [(y_L)_{bb} - (y_L)_b * (y_L)_b] \\
\partial R_L/\partial a &= e^{-Y_L} * [(y_L)_{ca} - (y_L)_c * (y_L)_a] \\
\partial R_L/\partial b &= e^{-Y_L} * [(y_L)_{cb} - (y_L)_c * (y_L)_b] \\
\partial R_L/\partial c &= e^{-Y_L} * [(y_L)_{cc} - (y_L)_c * (y_L)_c]
\end{aligned} \quad (22)$$

For samples subjected to right censoring, adjustments in terms of the derivatives of P_U, Q_U and R_U are required. These are given by the following equations:

$$\begin{aligned}
\partial P_U/\partial a &= P_U * [(y_U)_a * (e^{-Y_U} - 1) + (y_U)_{aa}/(y_U)_a - P_U] \\
\partial P_U/\partial b &= P_U * [(y_U)_b * (e^{-Y_U} - 1) + (y_U)_{ab}/(y_U)_a - Q_U] \\
\partial Q_U/\partial b &= Q_U * [(y_U)_b * (e^{-Y_U} - 1) + (y_U)_{bb}/(y_U)_b - Q_U] \\
\partial R_U/\partial a &= R_U * [(y_U)_a * (e^{-Y_U} - 1) + (y_U)_{ca}/(y_U)_c - P_U] \\
\partial R_U/\partial b &= R_U * [(y_U)_b * (e^{-Y_U} - 1) + (y_U)_{cb}/(y_U)_c - Q_U] \\
\partial R_U/\partial c &= R_U * [(y_U)_c * (e^{-Y_U} - 1) + (y_U)_{cc}/(y_U)_c - R_U]
\end{aligned} \quad (23)$$

The elements of the observed information matrix are obtained by adding the appropriate expressions in Eqs. 22 and 23 to those in Eq. 21 depending on whether the sample is left, right or doubly censored. These six second derivatives are used to form the 3x3 symmetric matrix needed in the iterative procedure.

Monte Carlo experiments and results

Generation of GEV variables

The inverse form of the GEV distribution with parameters a, b, c, is defined as:

$$x = c + (a/b) + [1 - (-\ln F)^b] \quad (24)$$

where F is the GEV distribution function.

The GEV variables can be obtained by converting the uniform random variables, U on (0,1), which were generated according to the algorithm developed by Wichmann and Hill (1982) by taking:

$$x = c + (a/b) * [1 - (-\ln U)^b] \quad (25)$$

Given a set of parameter values, the fixed censoring points are determined from:

$$x_L = c + (a/b) * [1 - (-\ln p)^b]$$
$$x_L = c + (a/b) * \{1 - [-\ln(1-q)]^b\} \quad (26)$$

where p and q are the truncation levels specified by the following probabilities:

$$p = \int_{-\infty}^{x_L} f(x)dx$$
$$q = \int_{x_U}^{\infty} f(x)dx \quad (27)$$

For each sample size N, N values of x were generated and censored at x_L and/or x_U, whereby the censored variable numbers m and/or k were introduced. The PWM and ML estimators were then obtained by using suitable equations.

TABLE 1
RELATIVE BIAS (PERCENTAGE) FOR UNCENSORED SAMPLES

N	Parameters a		b		c		X 100 years		500 years		1 000 years	
	PWM	ML	PWM	ML	PWM	ML	PWM	ML	PWM	ML	PWM	ML
30	0,8	2,6	-2,8	-15,3	-0,3	-1,3	-0,7	2,5	-2,1	2,0	-2,8	1,5
40	0,5	2,0	-2,3	-11,6	-0,2	-1,0	-0,5	2,2	-1,4	2,0	-1,9	1,9
50	0,4	1,6	-2,0	- 9,2	-0,2	-0,8	-0,3	1,8	-1,0	1,7	-1,4	1,6
60	0,3	1,4	-1,9	- 7,8	-0,2	-0,7	-0,2	1,6	-0,7	1,6	-1,0	1,6
70	0,2	1,1	-1,9	- 6,8	-0,2	-0,6	-0,1	1,4	-0,6	1,5	-0,9	1,4
80	0,1	1,0	-1,6	- 6,1	-0,1	-0,5	-0,1	1,3	-0,6	1,3	-0,8	1,3
90	0,1	0,9	-1,5	- 5,6	-0,1	-0,5	-0,1	1,2	-0,5	1,3	-0,7	1,3
100	0,1	0,8	-1,5	- 5,3	-0,1	-0,5	-0,1	1,1	-0,4	1,2	-0,6	1,2

TABLE 2
VARIANCES AND COVARIANCES OF PARAMETER ESTIMATORS FOR UNCENSORED SAMPLES

N	Var(a)		Var(b)		Var(c)		Cov(a,b)		Cov(b,c)		Cov(a,c)	
	PWM	ML	PWM	ML	PWM	ML	PWM	ML	PWM	ML	PWM	ML
30	39,93	39,15	0,020	0,025	78,02	77,59	0,45	0,55	0,52	0,61	3,28	7,77
40	28,96	28,47	0,014	0,015	56,80	57,07	0,32	0,36	0,36	0,40	2,49	4,77
50	23,18	22,19	0,011	0,011	45,71	45,48	0,25	0,27	0,28	0,30	1,58	3,24
60	19,11	18,81	0,009	0,009	38,16	37,34	0,21	0,22	0,23	0,23	1,54	2,43
70	16,35	15,92	0,008	0,007	32,52	32,26	0,18	0,18	0,20	0,20	1,49	2,00
80	14,30	13,89	0,007	0,006	28,38	27,53	0,16	0,16	0,18	0,17	1,18	1,73
90	15,60	12,54	0,006	0,005	25,40	24,74	0,14	0,14	0,16	0,15	1,08	1,46
100	11,38	10,95	0,005	0,005	22,76	21,98	0,13	0,12	0,14	0,13	0,97	1,31

TABLE 3
VARIANCES OF THE T-YEAR EVENT ESTIMATORS
(UNCENSORED SAMPLES)

N	Return period T 100 years		500 years		1 000 years	
	PWM	ML	PWM	ML	PWM	ML
30	575,8	558,4	1 383,6	1 179,8	1 884,4	1 493,6
40	418,9	335,9	963,2	689,0	1 283,6	866,6
50	330,1	259,6	746,2	534,5	985,8	674,1
60	269,4	226,7	602,7	460,5	719,6	579,5
70	230,5	171,7	511,4	353,7	668,8	447,4
80	200,9	151,1	443,7	309,6	578,4	391,2
90	176,5	125,4	386,1	255,3	501,3	322,6
100	157,4	115,9	343,3	177,8	445,1	250,0

Criteria used

In order to investigate the properties of parameter estimators and the effect of censoring on the estimation of the quantiles, the following statistics are adopted to be the performance evaluating indices:

- Mean of relative errors:

  The relative error of the parameter a, for example, is:

  $\mathrm{Rel}(a) = (a - \hat{a})/a$

  For a large value of NR, the number of replications, the mean of these relative errors is completed. This is more informative than the mean of the parameter estimator, since the bias of the estimation has been adjusted by the magnitude of the parameter itself. The mean of relative errors of parameter therefore can be considered as the relative bias in estimating that parameter.
- Variances and covariances:

  The simulated variances and covariances of parameter estimators can be computed directly from 10 000 replications. Updating formulas (Phien, 1988) were used in this connection.
- T-year event:

  The T-year event x_T is defined as

  $$\mathrm{Prob}(x > x_T) = 1/T$$

  and in view of Eq. 24:

  $$X_T = c + (a/b) + \{-[-\ln(1 - 1/T)]^b\} \quad (28)$$

  The relative bias in estimating X_T is the mean of the relative error $\mathrm{Rel}(X_T) = (X_T - \hat{X}_T)/X_T$

  where $\hat{X}_T$ is an estimate of X_T.

In practical applications, the variance of the T-year event is more important than the variances of the parameter estimators. It can be directly computed from the estimated values of X_T in NR replications.

In this study, for any sample size N = 30(10)100, 10 000 replications (NR = 10 000) were generated on the ND-570 available at the Division of Computer Science, Asian Institute of Technology. For each sample and in each replication, the PWM and ML were used to estimate the parameters of the GEV. Once the parameters have been estimated, the corresponding value of X_T can be computed according to Eq. 24 with the estimated values substituted for a, b and c. As much work has been done for the EV1 (a particular case of the GEV with b = 0) (Phien, 1986; 1987), Raynal and Salas (1987), this study focused on the GEV only. The results reported in the following were obtained for:

$a = 42{,}5 \qquad b = 0{,}2486 \qquad c = 105{,}8$

but the same pattern was also observed for other sets of parameter values. The values used for T are 100, 500 and 1 000 years, that correspond to the following non-exceedance probabilities:

0,99 0,998 and 0,999

Correspondingly:

$X_{100} = 222{,}87 \qquad X_{500} = 240{,}28 \qquad X_{1\,000} = 246{,}57$

Results and discussions

For uncensored samples, the relative bias of each parameter and each T-year event is shown in Table 1. It is clear from this Table that:

- the relative bias for both the PWM and ML methods decreases with an increasing sample size;
- the PWM estimators have a lower relative bias than the corresponding ML estimators; and
- as the non-exceedance probability increases, i.e. T increases, the PWM estimator of X_T become slightly more biased, while the ML estimator does not have a clear pattern.

Since all the above relative biases are small, one may consider both PWM and ML estimators to be unbiased for N = 30(10)100.

Also for uncensored samples, the PWM seems to produce values of the variances/covariances of the estimators quite close to those produced by the ML (Table 2). For the T-year event, however, the variance estimated by the PWM is much larger than that estimated by the ML (Table 3). The difference between the two methods becomes more distinct with an increase in the values of T, i.e. in the upper tail of the GEV distribution.

For censored samples, the PWM produces unreasonable results in terms of both the relative bias and the variances of the T-year event. Extremely large values were observed quite frequently for both these statistics. Such an inferior performance of the PWM deserves no further considerations.

The problem now reduces to the investigation of the effect of censoring on the ML estimators. To save space, the relative bias and the variance of the T-year event estimator are tabulated for a number of censoring levels.

- From Table 4, it is seen that the relative bias in estimating the T-year event decreases as the sample size increases in the same way experienced from uncensored samples (Table 1), although several irregularities exist. It seems that, for this particular set of parameter values, censoring on the left at low level ($\leq 0{,}10$) can reduce the bias. Censoring to the right and double censoring may reduce the bias on large samples.
- From Table 5, it is clear that the variances of the T-year event estimator for censored samples increases in most cases. However, for a sample of 100 (or 30), left (or right) censoring may slightly reduce the variance.

From the simulation results just reported, it is clear that the effect of censoring on the ML estimators depends on whether it is left, right or double censoring. In other words, censoring at the same level (when the censoring points are represented by the corresponding probabilities of non-exceedance for left censoring and exceedance for right censoring) can result in different effects. This difference is believed to be due to the non-symmetry of the GEV distribution.

The fact that the ML method provides reasonable estimators for the parameters and the quantiles (expressed in the form of the T-year event) of the GEV is quite encouraging in exploring the capability of the Akaike information criterion (AIC) (Akaike, 1974) in identifying an extreme distribution from the three extremal models (Turkman, 1985).

TABLE 4
RELATIVE BIAS (PERCENTAGE) OF THE ML ESTIMATOR OF THE T-YEAR EVENT

p	q	T	Sample size, N 30	40	50	60	70	80	90	100
0,05	0,0	100	0,9	1,1	0,9	0,7	0,7	0,7	0,7	0,6
		500	− 0,9	0,2	0,1	0,1	0,2	0,3	0,3	0,2
		1 000	− 1,9	−0,4	−0,3	−0,3	−0,1	0,1	0,0	0,0
0,0	0,05	100	1,7	2,2	2,2	2,4	2,4	2,1	2,1	2,0
		500	0,2	1,6	1,9	2,4	2,4	2,2	2,3	2,1
		1 000	0,6	1,1	1,6	2,2	2,4	2,2	2,3	2,2
0,025	0,025	100	− 4,6	−2,5	−1,6	−1,1	−0,8	−0,6	−0,4	−0,4
		500	− 9,1	−4,9	−3,3	−2,3	−1,7	−1,3	−1,0	−0,9
		1000	−11,6	−6,1	−4,1	−3,0	−2,2	−1,7	−1,4	−1,2
0,10	0,0	100	1,2	1,5	1,2	1,1	0,8	0,9	0,9	0,7
		500	− 0,5	0,7	0,5	0,6	0,3	0,5	0,7	0,4
		1 000	− 1,5	0,2	0,1	0,3	0,0	0,3	0,5	0,2
0,0	0,10	100	4,0	3,4	2,7	2,3	1,6	1,3	1,1	0,8
		500	3,2	2,9	2,3	2,0	1,7	1,0	0,8	0,5
		1 000	2,6	2,5	2,1	1,8	1,1	0,8	0,7	0,3
0,05	0,05	100	− 5,7	−3,0	−2,2	−1,5	−1,2	−1,0	−0,8	−0,7
		500	−12,2	−6,3	−4,5	−3,3	−2,5	−2,1	−1,7	−1,6
		1 000	−16,2	−8,1	−5,7	−4,1	−3,1	−2,7	−2,2	−2,0

TABLE 5
VARIANCE OF THE ML ESTIMATOR OF THE T-YEAR EVENT

p	q	T	Sample size, N 30	40	50	60	70	80	90	100
0,05	0,0	100	519,6	331,4	285,7	202,9	180,9	159,9	127,3	106,4
		500	1 157,7	725,6	618,8	436,8	388,2	340,8	271,7	223,1
		1 000	1 498,1	936,3	796,2	562,4	499,4	437,3	349,5	285,9
0,0	0,05	100	450,2	341,5	303,3	228,2	221,9	168,8	180,8	162,7
		500	1 007,9	742,8	646,3	482,5	463,1	354,1	376,8	336,3
		1 000	1 299,8	949,3	820,9	612,0	584,5	448,5	475,7	423,7
0,025	0,025	100	759,1	486,3	404,1	344,5	285,4	220,5	203,9	191,1
		500	1 773,5	1 094,8	896,5	756,4	623,7	479,9	440,5	414,6
		1 000	2 333,4	1 422,4	1 157,7	973,1	801,0	616,1	563,5	531,8
0,10	0,0	100	560,6	335,8	243,9	207,7	199,0	163,6	136,9	102,9
		500	1 246,9	718,3	517,0	437,0	417,3	344,5	282,7	212,0
		1 000	1 613,6	921,9	663,4	558,8	532,7	440,4	359,9	270,3
0,0	0,10	100	549,2	461,8	401,5	343,7	341,3	287,7	270,2	232,9
		500	1 139,8	947,4	824,2	703,5	700,5	592,9	553,4	483,0
		1 000	1 430,1	1 186,1	1 032,7	881,8	879,2	745,7	695,1	609,4
0,05	0,05	100	1 067,4	745,1	543,6	453,2	402,1	354,3	280,4	253,1
		500	2 472,6	1 674,7	1 221,9	1 011,3	890,3	779,3	619,2	559,4
		1 000	3 228,3	2 164,1	1 578,4	1 303,1	1 144,1	999,5	795,6	718,8

Akaike information criterion

This criterion can be stated as follows:

$$AIC = -2L + 2r$$

where L denotes the log-likelihood function and r is the number of parameters involved. For the EV1, $r = 2$, while for the other two remaining models EV2 and EV3, $r = 3$.

In the simulation, the number of replications for each sample size $N = 30(10)100$ was 1 000. Two sets of parameter values were used:

I : a = 42,5 b = 0,2486 c = 105,80 (EV3)
II: a = 42,5 b = − 0,2486 c = 105,80 (EV2)

The percentage of times that the AIC identifies the model correctly are collected in Table 6. It is clear that the AIC can identify the extremal models with a very high accuracy level. In all cases considered over 90 per cent of the time, the identification is correct.

TABLE 6
CORRECT IDENTIFICATION BY AIC EXPRESSED IN PERCENTAGE

N	(I)				(II)			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
30	97,6	95,4	96,6	89,9	92,2	92,9	92,2	91,9
40	99,1	98,1	97,1	95,6	95,7	96,2	95,2	94,7
50	99,3	98,7	98,5	96,8	97,0	97,8	96,2	95,9
60	99,7	99,6	99,2	97,7	98,5	99,2	98,4	97,8
70	99,9	99,6	99,8	99,1	99,4	99,2	99,2	98,5
80	99,9	100,0	99,9	99,5	99,4	99,4	99,4	99,1
90	100,0	100,0	100,0	99,5	99,5	99,8	99,4	99,6
100	100,0	100,0	100,0	99,7	99,5	99,6	99,4	99,5

(I) a = 42,5 b = 0,2486 c = 105,80
(II) a = 42,5 b = − 0,2486 c = 105,80
(1) uncensored samples
(2) left censoring p = 0,05 q = 0
(3) right censoring p = 0 q = 0,05
(4) double censoring p = q = 0,025

Use of the AIC has a clear advantage over some existing statistical tests like that of Hosking (1984) or the likelihood-ratio test, because no significance level is to be specified.

Summary and conclusions

The estimation of the parameters of the general extreme value (GEV) distribution by the methods of probability-weighted moments (PWM) and maximum likelihood (ML) was treated in this study for both complete and type I censored samples. The PWM was extended to the case of censored samples by adjusting the ranks of the elements in the uncensored portion with respect to the whole sample. All the needed expressions were derived and simulation experiments were carried out to evaluate the performance of the two methods. Finally the AIC was applied to identify the suitable extremal model from the GEV for both uncensored and censored samples. From the simulation results, the following conclusions can be drawn:

- For uncensored samples, the PWM produces estimators with less bias than the ML. However, the bias is quite small for both methods. In terms of the variances/covariances of the parameter estimators, the PWM seems to have a comparable efficiency as the ML. The variance of the extreme quantile, like the T-year event with T > 100, estimated by the PWM is, however, quite large compared to that estimated by the ML.
- For censored samples, the PWM (in the present extension) does not perform satisfactorily in terms of the bias and variance. It was dropped from further considerations related to censored samples.
- Censoring at a certain level may reduce the bias of the ML estimators, and it might reduce even the variances of these estimators in some cases.
- The AIC (used along with the ML method) can identify the extremal model with a high accuracy level, for both uncensored and censored samples.

At present, the work on the determination of the elements of Fisher's information matrix and its approximation to the inverse of the variance-covariance matrix is being undertaken. The results of this work will soon be reported.

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