Changes in water table induced by transient boundary condition and space-dependent recharge - A Green element approach

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Abstract

An efficient boundary integral procedure has been developed to study groundwater fluctuations for a well-aquifer system characterised by a transient boundary condition. This technique, referred to as the Green element method (GEM) is a hybrid procedure which implements the Boundary element method (BEM) in such a way that domain integration is carried out element by element. By adopting a procedure that is close to the Finite element method (FEM), GEM is able to yield a solution that not only agrees with the physics of the problem and the accompanying boundary conditions, but compares favourably with an available closed form solution.

Introduction

The problem of groundwater movement through an underlying porous medium is very important in several aspects of engineering. Groundwater movement plays a role in agricultural, environmental and industrial processes. In all these cases, the engineer is primarily concerned with the problem of quantifying or determining the net movement of water due to losses and various means of transport. Efficient management of aquifers requires an accurate estimation of sustainable yield at which groundwater can be exploited without stressing the porous medium. Recently, this concern has acquired new dimensions due to rising concerns about continued and sometimes intensive groundwater exploitation in areas where surface water is becoming insufficient.

The flow of water through porous media is a subject of great interest and has attracted a number of researchers. Early investigations on groundwater resources of the African Sahara were performed by Hammad (1969) on a series of wells arranged in line with small spaces in between them. Some of his solutions could not be related to the boundary and initial conditions proposed. Later, Gill (1981) improved on the solution of Hammad (1969) by providing a new set of initial and boundary conditions and formulated a transient flow problem whose solution reduced to a steady state after a long period of time. Mustafa (1984), working on the same problem as Gill (1981) introduced a leakage and a recharge source and applied Fourier series analysis to arrive at time-dependent solutions of his transient problem. Ram et al. (1994) assumed recharge as a line source in addition to a vertical infiltration. Their solutions agreed closely with those presented by Mustafa (1984). Onyejekwe (1994) arrived at an analytic solution to the problem of an unsteady flow to an observation well from a semi-confined leaky aquifer by converting both the non-homogeneous governing equation and the boundary conditions to a set of Sturm-Liouville problems. Further work on a semi-confined leaky aquifer subjected to a spatially varying recharge and transient boundary condition was presented by Onyejekwe (1998). Plausible agreements were achieved on comparison of his results with those obtained by Gill (1981) and Mustafa (1984). Analytic solutions were also presented by Mustafa (1987), Marino (1974) and Latinopoulos (1981) for various cases of groundwater flow through porous media.

In the work reported herein, we adopt the Green element method (GEM) to determine changes in the ground water table by comparing our numerical results with those obtained by Onyejekwe (1998).

Problem formulation

The partial differential equation which describes the movement of water in a confined aquifer coupled with recharge is given by (Mustafa, 1984):

$$\nu\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} + \mu = 0, \qquad on\ x_o \le x \le x_L \tag{1}$$

where:

μ (the source term) = *x(1-x)*
$u = u(x,t)$, is the height of water table
$v = T/S$
T = aquifer transmissivity
S = the storage coefficient
$L = x_0 - x_L$ is the length of the problem domain.

For Eq. (1) to be well posed, information is required at initial time (t = 0) and at the boundaries.

We propose a Green element numerical solution of Eq. (1). Most of the theoretical and computational aspects involved in the development of GEM can be found in our previous papers (Onyejekwe, 1996; Taigbenu and Onyejekwe, 1997; Onyejekwe, 1997), and as such no effort will be made to go through them in detail. As indicated earlier, GEM is based essentially on the boundary integral theory, but its element-by-element implementation follows that of the Finite element method (FEM). The resulting hybrid procedure yields a method that is more adapted to handle those problems which give rise to considerable numerical difficulty to Boundary element method (BEM) (Taigbenu and Onyejekwe, 1997; Onyejekwe, 1996)). We, however, hasten to

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comment that the whole idea of GEM is not to replace other **"traditional"** and more **"entrenched"** numerical techniques, but to enhance the numerical application of the boundary integral theory to the solution of engineering problems. In this context, we test the suitability of GEM on a groundwater problem involving a transient boundary specification, and a recharge function.

Application of GEM to solve Eq. (1) essentially requires the following steps:

- Integral replication of the governing partial differential equation.
- Discretisation and representation of the resulting integral equation on a generic element of the problem domain.
- A finite element type solution to determine the field variables.

We start by converting Eq. (1) into its integral form. This is achieved more straightforwardly via the Green's second identity. For two functions *G* and *u* which are twice differentiable, the Green's second identity is formally represented as:

$$\int_{x_o}^{x_L} \left[u\frac{\partial^2 G}{\partial x^2} - G\frac{\partial^2 u}{\partial x^2} \right] dx = \left[u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x} \right]_{x=x_0}^{x=x_L} \quad (2)$$

In order for us to be able to utilise, Eq. (2), we seek a complementary differential equation to Eq. (1). This is given by:

$$\frac{\partial^2 G}{\partial x^2} = \delta\ (x-x_i) \quad (3)$$

where $\delta(x-x_i)$ is the Dirac delta function. Let the solution of Eq. (3) be of the form:

$$G(x,x_i) = \frac{1}{2}(|x-x_i|+k) \quad (4)$$

where *k* is an arbitrary constant, and $G(x,x_i)$ is known as the fundamental solution. It can be physically interpreted as the response of a system governed by Eq. (3), when it is subjected to a unit instantaneous input. The derivative of Eq. (4) with respect to *x* is given by:

$$\frac{dG(x,x_i)}{dx} = \frac{1}{2}\ [H(x-x_i) - H(x_i-x)] \quad (5)$$

where $H(x,x_i)$, the Heaviside function, is defined as:

$$H(x,x_i) = \begin{cases} 1, & x > x_i \\ 0, & x < x_i \end{cases} \quad (6)$$

Introducing Eqs. (1) and (3) into Eq. (2) yields:

$$\int_{x_o}^{x_L} \left[u\delta(x-x_i) - \frac{G}{\nu}\left(\frac{\partial u}{\partial t} + \mu \right) \right] dx = \left[u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x} \right]_{x=x_0}^{x=x_L} \quad (7)$$

Eq. (7) can now be simplified to give:

$$-\lambda u(x_i,t) + \left[u\frac{\partial G}{\partial x}\ G\frac{\partial u}{\partial x} \right]_{x=x_0}^{x=x_L} + \int_{x_o}^{x_L} \frac{G}{\nu}\left(\frac{\partial u}{\partial t} + \mu \right) dx = 0 \quad (8)$$

in which λ takes the value of unity if the source point x_i is within the computational domain or half if it is located at the boundaries. Eq. (8) is the integral representation of the governing partial differential equation which is solved for each element of the problem domain to yield both the primary variable and its derivative. The hybridisation procedure resumes with the division of the problem domain into elements. If for example we have a total of M_e number of elements, Eq. (8) can be written for all the elements as:

$$\sum_{e=1}^{M_e} -2\lambda^{(e)}u_i^{(e)} + [H(x_2^{(e)} - x_i^{(e)}]u_2^{(e)} -$$
$$[H(x_1^{(e)} - x_i^{(e)} - H(x_i^{(e)} - x_1^{(e)}]u_1^{(e)} - (|x_2^{(e)} - x_i^{(e)}| + k)\phi_2^{(e)} +$$
$$(|x_1^{(e)} - x_i^{(e)} + k)\phi_1^{(e)} + \frac{1}{\nu}\int_{x_1^{(e)}}^{x_2^{(e)}} [(|x - x_i^{(e)}| + k)\left(\frac{\partial u}{\partial t} + \mu \right)]dx = 0 \quad i=1,2 \quad (9)$$

where the length of a typical element is given by $l^{(e)} = x_2^{(e)} - x_1^{(e)}$, and x_2, x_1 are the co-ordinates of the end points.

Numerical procedure

Having described the theoretical framework of GEM, the basic concepts of its numerical implementation are now discussed. Unlike BEM, domain discretisation is not considered a disadvantage; this aspect of FEM lends the GEM much of its robustness and versatility (Onyejekwe, 1996). In addition to dividing the problem domain into elements, the dependent variables are specified in terms of the summation of the products of the interpolation functions and the unknown nodal values of the field variable.

$$u(x,t) = \Omega_1^{(e)}(x)u_1^{(e)}(t) + \Omega_2^{(e)}(x)u_2^{(e)}(t)$$
$$\phi(x,t) = \frac{\partial u(x,t)}{\partial x} = \Omega_1^e(x)\frac{\partial u_1^e(x,t)}{\partial x} + \Omega_2^e(x)\frac{\partial u_2^e(x,t)}{\partial x} \quad (10)$$

where:

$\Omega_1^{(e)} = 1 - \xi$, $\Omega_2^{(e)} = \xi$ are interpolating functions

$\xi = (x-x_1^{(e)})/l^{(e)}$ is a local co-ordinate.

Substituting Eq. (10) into Eq. (9) yields:

$$\sum_{e=1}^{M_e} \nu(-2\lambda_i^{(e)}u_i^{(e)} + [H(x_2^{(e)} - x_i^{(e)}) - H(x_i^{(e)} - x_2^{(e)}]u_2^{(e)} -$$
$$[H(x_1^{(e)} - x_i^{(e)}) - H(x_i^{(e)} - x_1^{(e)}]u_1^{(e)}\ (|x_2^{(e)}\ x_i^{(e)}| + \bar{l})\phi_2^{(e)} + (|x_1^{(e)} - x_i^{(e)}| + \bar{l})\phi_1^{(e)} +$$
$$\int_{x_1^{(e)}}^{x_2^{(e)}} (|x - x_i^{(e)}| + \bar{l}) \left[\Omega_1^{(e)}\ (\frac{du_1^{(e)}}{dt} + \mu_1^{(e)}) + \Omega_2^{(e)}\ (\frac{du_2^{(e)}}{dt} + \mu_2^{(e)}) \right] dx = 0 \quad (11)$$

Observe that we have substituted the longest spatial element size in the problem domain for the arbitrary constant *k*; this guarantees that the coefficient matrix remains positive. We obtain two equations from Eq. (11), by considering the positions of the source node at x_1 and x_2 respectively. If the source node x_i is located at x_1 we obtain a discretised equation given by:

$$\sum_{e=1}^{M_e} \nu(-u_1^{(e)} + u_2^{(e)} + \bar{l}\phi_1^{(e)} - (\bar{l} + l^{(e)})\ \phi_2^{(e)}) +$$
$$l^{(e)} \int_0^1 (\bar{l} + l^{(e)}\zeta) \left[\Omega_1^{(e)}\ (\frac{du_1^{(e)}}{dt} + \mu_1^{(e)}) + \Omega_2^{(e)}\ (\frac{du_2^{(e)}}{dt} + \mu_2^{(e)}) \right] d\zeta = 0 \quad (12)$$

Similarly when the source node is at x_2, the following discretised equation applies:

$$\sum_{e=1}^{M_e} \nu\left(+u_1^{(e)}-u_2^{(e)} + (\bar{l} + l^{(e)})\phi_1^{(e)} - \bar{l}\phi_2^{(e)}\right) +$$

$$l^{(e)} \int_0^1 \left(\bar{l}+l^{(e)}(1-\zeta)\right) \left[\Omega_1^{(e)} \left(\frac{du_1^{(e)}}{dt} + \mu_1^{(e)}\right) + \Omega_2^{(e)} \left(\frac{du_2^{(e)}}{dt} + \mu_2^{(e)}\right) \right] d\zeta = 0 \qquad (13)$$

As a result of this procedure, we have discretised the governing partial differential equation and replaced it with a set of quasi-ordinary differential equations at the nodes. These equations can now be put into a single matrix equation given by:

$$\sum_{e=1}^{M_e} DR_{ij}^{(e)}u_j^{(e)} + DL_{ij}^{(e)}\phi_j^{(e)} + T_{ij}^{(e)}\left[\frac{du_j^{(e)}}{dt} + \mu_j\right] = 0 \qquad (14)$$

and the elements of the coefficient matrix are given by:

$$R_{i,j}^{(e)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = (-1)^{i+j-1} \; ; \qquad i,\, j = 1,\, 2 \qquad (15)$$

$$L_{i,j}^{(e)} = \begin{bmatrix} \bar{l} & -\bar{l}+l^{(e)} \\ \bar{l}+l^{(e)} & -\bar{l} \end{bmatrix} ; \; i,\, j = 1,\, 2 \qquad (16)$$

$$T_{ij} = \begin{bmatrix} 3\bar{l}+l^{(e)} & +3\bar{l}+2l^{(e)} \\ 3\bar{l}+2l^{(e)} & 3\bar{l}+2l^{(e)} \end{bmatrix} \; ; \; i,j=1,2 \qquad (17)$$

Approximation in time

Various numerical techniques can be adopted for the time derivative of Eq. (14). In the work reported herein, we shall adopt a 2-level time discretisation scheme to approximate the temporal derivative. This procedure carries out the approximation of the time derivative at $t = t_{m+\alpha} = t_m + \alpha\Delta t$. This procedure is given by the following finite difference expression:

$$\frac{du_j}{dt}\Big|_{t=t_m+\alpha\Delta t} \approx \frac{u_j^{(e)}(t_m+\Delta t)-u_j^{(e)}(t_m)}{\Delta t} = \frac{u_j^{(e,m+1)}-u_j^{(e,m)}}{\Delta t} \; ; \quad 0 \le \alpha \le 1 \qquad (18)$$

where t_m is the previous time level, $\Delta t = t_{m+1} - t_m$ is the time step, and t_{m+1} is the current time level where the solution is determined, and α is a weighting factor which varies from 0 to 1, and positions the time derivative at the time level which it will be evaluated. Conventional definitions of these time levels in the finite difference and finite element methodologies describe $\alpha = 1.0$ as the fully implicit scheme, $\alpha = 0$ as the fully explicit scheme, $\alpha = 0.5$ is referred to as the Crank-Nicholson sheme, and $\alpha = 0.67$ is given as the Galerkin's scheme. In this work, The Galerkin's sheme gives results that are closest to the closed form solutions. Since the time term has been evaluated at $t_m + \alpha\Delta t$, we can now go ahead and evaluate other terms of Eq. (14) at this time level. We then adopt a weighted average expression given by:

$$\sum_{e=1}^{M} \alpha(\nu R_{i,j}^{(e)}u_{j,m+1}^{(e)} + \nu L_{i,j}^{(e)}\phi_{j,m+1}^{(e)}) + (1-\alpha)(\nu R_{i,j}^{(e)}u_{j,m}^{(e)} + \nu L_{i,j}^{(e)}\phi_{j,m}^{(e)}) +$$

$$T_{i,j}^{(e)}\left[\frac{1}{\Delta t}\,(u_{j,m+1}^{(e)} - u_{j,m})^{(e)} + (\alpha\mu_{j,m+1}^{(e)} + (1-\alpha)\mu_{j,m}^{(e)})\right] = 0 \quad ; 0 \le \alpha \le 1 \qquad (19)$$

The subscripts m and $m+1$ refer to the value of the coefficients at time t and $t + \Delta t$ respectively. A global system of matrix equation representing the problem is given by:

$$[A]_{i,j} \begin{pmatrix} u_j \\ \phi_j \end{pmatrix} = (RHS_i) \qquad (20)$$

where A is the coefficient matrix, and RHS is the right-hand vector. Eq. (20) is a linear algebraic equation and can be solved by any method based on Gaussian elimination.

Example problem and discussion

With GEM formulation completed, we now compare our numerical solution to the analytical solution obtained by Onyejekwe et al. (1997). The example problem is a leaky confined aquifer of length L resting on a horizontal boundary bounded by a well on one side and a river on the other as shown in Fig. 1. The aquifer considered is assumed to be homogeneous with respect to storage and isotropic with respect to transmissivity. The water level in the well is assumed to be $U_0 + S_o$ initially and drops to U_0 after a long period of time ($U_0 = 1$, $S_o = 0.5$ and transmissivity, $T = 1$, $\nu = 1$ for this problem). The height of the water in the river is assumed to be constant and that at the well varies as a function of time. In addition a parabolic recharge function $R = x(1-x)$ is imposed on the system. The problem thus specified is desribed by the following initial and boundary conditions:

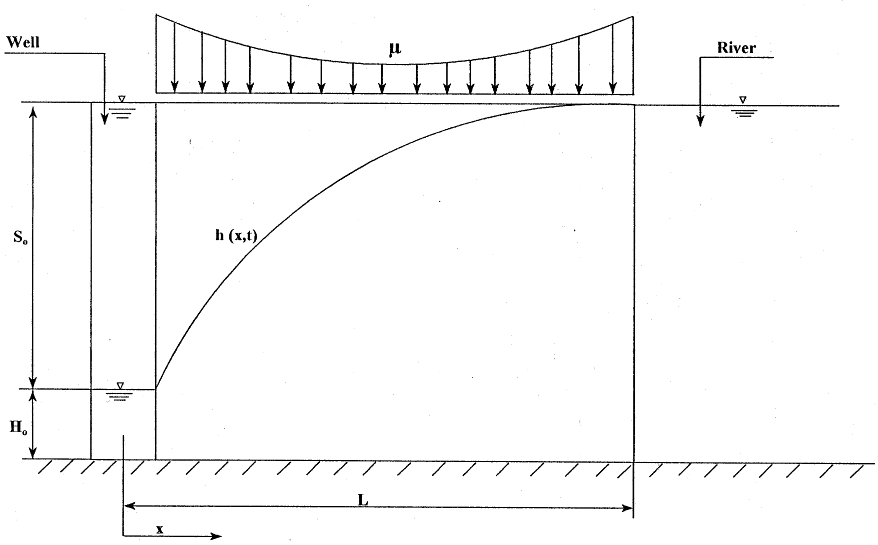

Figure 1
Schematic representation of the problem

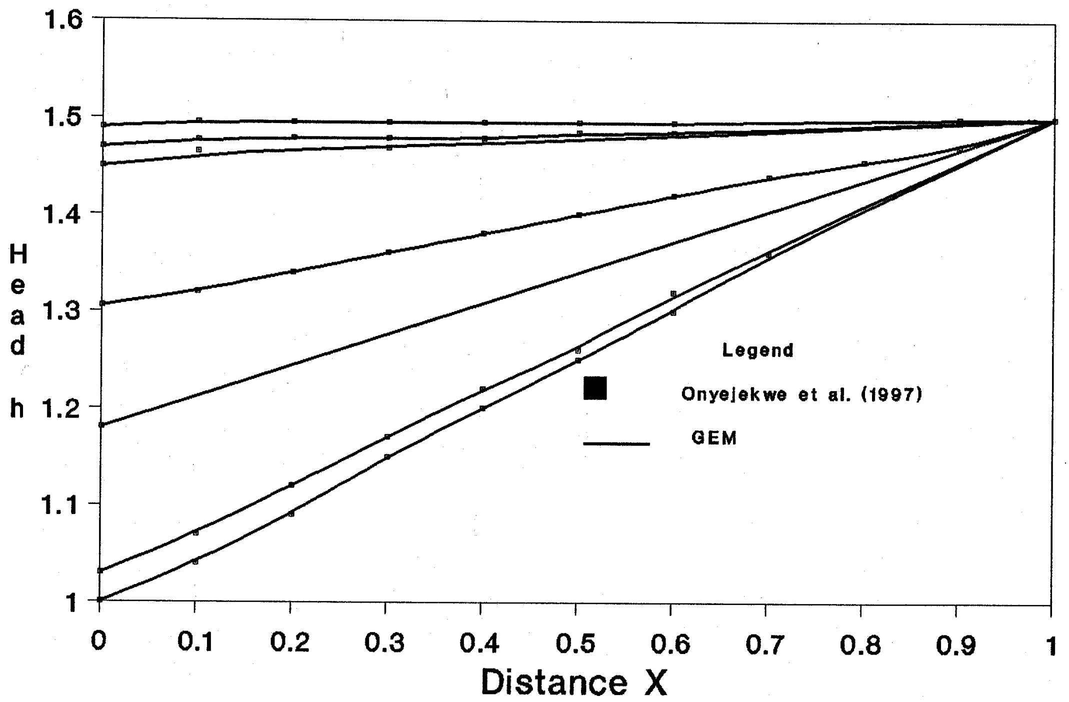

Figure 2
Variation of head with distance

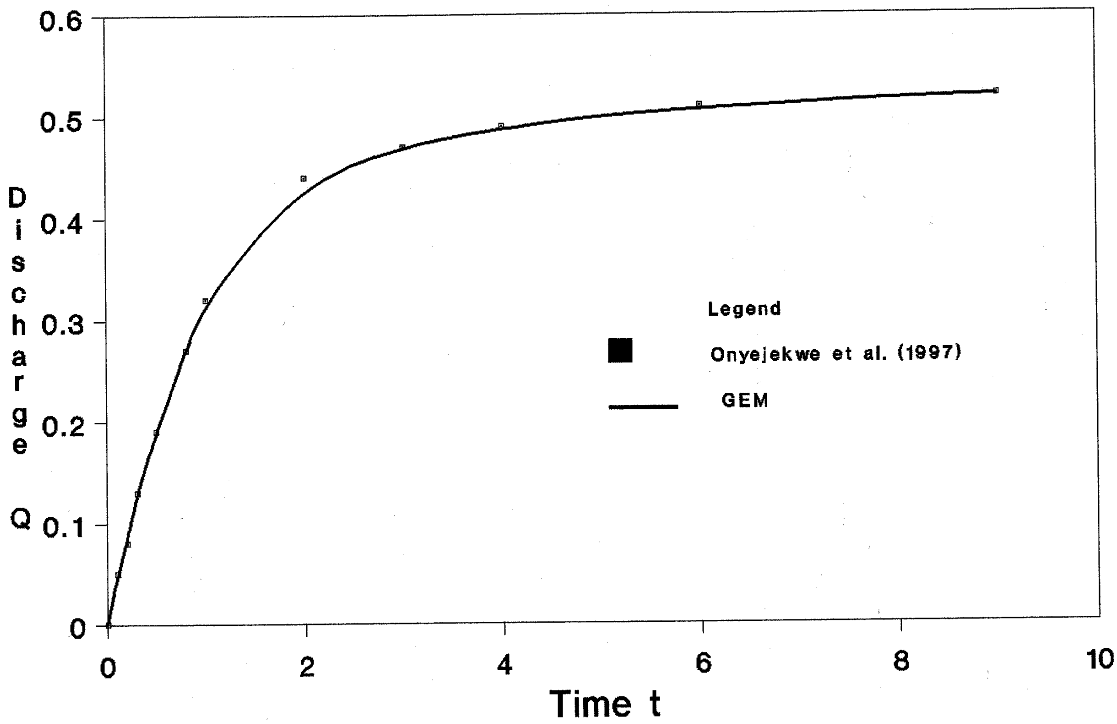

Figure 3
Variation of discharge with time

Initial condition: $u(x,0) = U_0 + S_0$

Boundary conditions: $u(0,t) = U_0+S_0e^{-Pt}$ t>0
$u(L,t) = U_0+S_0$ t>0 (21)

where for this problem, P the time coefficient is unity , the time step $\Delta t = .01$ and the total number of elements $M_e = 10$. The closed form solution to this problem as presented by Onyejekwe (1998) is:

$$u(x,t) = U_o + S_o e^{-Pt} + \frac{S_o}{L} x(1 - e^{-Pt}) + \sum_{n=1}^{\infty} A_n(t)\, e^{\frac{-\nu n^2\pi^2 t}{L^2}} \sin(\frac{n\pi}{L} x) \quad (22)$$

where $A_n(t)$ is given by:

$$A_n(t) = \frac{PS_o\,[\,e^{-Pt} - e^{-\lambda_n^2 t}\,]\,[\,\frac{1}{\lambda_n} - \frac{1}{L\lambda_n^2}\sin(\lambda_n L)\,]}{[\,\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}\,]\,[\,\lambda_n^2 - P\,]} + \frac{[\,1 - e^{-\lambda_n^2 t}\,]\,[\,\cos(\lambda_n L)(\frac{L^2}{\lambda_n} - \frac{L}{\lambda_n} - \frac{2}{\lambda_n^3})\,]}{\lambda_n^2\,[\,\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}\,]} + \frac{[\,1 - e^{-\lambda_n^2 t}\,]\,[\,\sin(\lambda_n L)(\frac{1-2L}{\lambda_n^2}) + \frac{2}{\lambda_n^3}\,]}{\lambda_n^2\,[\,\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}\,]} \quad (23)$$

where:

λ_n = $n\pi/L$ are eigenvalues corresponding to the Fourier decomposition of the solution.

Using Darcy's equation, the net flow rate in the ditch can be determined from:

$$Q = T\,(\frac{\partial h}{\partial x})_{x=0} \quad (24)$$

Differentiating Eq. (24) with respect to *x* and setting *x* to be zero, the rate of flow is:

$$Q = \frac{TS_o}{L}(1 - e^{-PT}) + \sum_{n=1}^{\infty} T\,(\frac{n\pi}{l})\,A_n(t)\,e^{\frac{-\alpha n^2\pi^2 t}{L^2}} \quad (25)$$

Results and conclusions

Results obtained by GEM and those presented by Onyejekwe (1998) are shown in Figs. 2 and 3. It can be noted that the two plots are in agreement with the physics of the problem and the specified initial and boundary condition. In Fig. 2 when *x=0,* i.e. at the well, the height of water was found to decrease with increase in time. After a long time interval, this value was found to be constant at U_0. At x=L, the height of water in the river remains fixed at $U_0 + S_o$. Similarly for Fig. 3, there is a steep rise in flux for values of t less than 3, but for longer periods, the flow rate stabilises as indicated by Eq. (25).

A boundary integral procedure has been developed for determining the changes in water table exposed to a transient boundary condition and space-dependent recharge. This technique was compared with the closed form solution obtained by Onyejekwe (1998) and excellent results were obtained. Extension of GEM to 2-D application is straightforward (Taigbenu and Onyejekwe, 1995).

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