A modified boundary integral solution of recharging and dewatering of an unconfined homogeneous aquifer

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Abstract

A numerical procedure is presented to deal with recharging and dewatering of unconfined homogeneous aquifer. The new procedure, referred to as the Green element method (GEM), relies on the singular integral theory of the boundary element method (BEM) and permits the solution integral replication of the governing non-linear partial differential equation by domain discretisation. Hence, the advantages of both BEM and the finite element method (FEM) are utilised. Numerical results from test problems obtained by GEM are compared with those from the literature, and for all cases, the results were found to be very encouraging.

Introduction

World-wide, the demand for water has greatly increased while the available amount of water is limited. As a result, reclamation and reuse of water resources has received a great amount of attention in recent times. This has necessitated the need to explore new methods of exploiting water, especially water in aquifers. One of the earlier studies on unconfined aquifers was made by Marino and Yeh (1972). They considered a recharge well with unsteady radial flow in an unconfined homogeneous and isotropic aquifer together with a source term and solved the governing partial differential equation by a method involving transformation and Lagrange interpolation. Their results were very close to the analytical solution of the problem under consideration. Marino (1973) developed analytical expressions that describe the water fluctuations in semi-pervious stream-unconfined aquifer systems. He considered the water level in the stream to be lowered suddenly below its initial elevation and then suddenly raised above its initial elevation while the storage capacity of the stream bed remained insignificant. The only setback to his work was that the expression derived are applicable only when the rise or decline of the water table does not exceed 50% of the initial depth of saturation. Further work on unconfined aquifer done by Marino (1975) on water table fluctuations beneath a circular uniformly recharging area resulted in a numerical solution based on the Douglas-Jones Predictor-corrector method. His solution gave encouraging results when compared with those available in literature. Rao and Sarma (1984) developed an analytical solution for determining a groundwater profile resulting from localised recharge to a finite unconfined aquifer with mixed boundary conditions. In their study, they used the extended finite Fourier transforms and the method of images to arrive at the analytical solution. Their analytical results were validated by experimental results. Latinopoulos (1981) presented an analytical solution for groundwater flow in an unconfined aquifer under seasonal recharge. He showed that to have a uniform recharge rate over the whole period does not enhance an accurate analytical solution. Another contribution was made by Lockington (1997) in determining the water table in an unconfined aquifer bounded by a stream. He considered the aquifer dewatering and recharging. In his work, he presented analytical solutions of the Boussinesq equation that describes the recharging and dewatering process in an unconfined aquifer.

The scarcity of published work in groundwater literature concerning the use of the boundary element method (BEM) to solve the non-linear Boussinesq equation that describes the flow of moisture in an unconfined aquifer is mainly due to the numerical difficulties encountered in applying this method to resolve non-linearity. In the work reported herein, we adopt a novel numerical procedure (Onyejekwe 1995; Taigbenu and Onyejekwe, 1997) based on the boundary integral theory to resolve non-linearity in an efficient and straight-forward way. This is the key motivation for this work.

Problem formulation

The non-linear Boussinesq equation that describes the flow of water in an unconfined aquifer obtained under Dupuit-Forchheimer assumptions is given by :

$$\frac{K}{S}\frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right) = \frac{\partial\phi}{\partial t} + \frac{R(x,t)}{S} \qquad (1)$$

where:

ϕ *(x,t)* is the height of water table above impervious layer
x and t are space and time co-ordinates respectively
K is the hydraulic conductivity
S is the specific yield
R(x,t) is the source-sink term.

For Eq. (1) to be well posed, appropriate initial and boundary conditions should be specified. The initial condition for Eq. (1) that describes the distribution of the scalar variable at initial time for a computational domain, Ω, is given by:

$$\phi(x,t_o) = \phi_o \quad \text{on } \Omega \qquad (2)$$

The Dirichlet data that can be specified on a portion of the

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boundary Γ_1 are given as:

$$\phi(x,t) = \phi_1 \quad \text{on } \Gamma_1 \; t > t_o \tag{3}$$

and the Neumann or flux boundary condition can be specified on another part of the boundary as:

$$\phi\partial \; \phi/\partial n = q_n \quad \text{on } \Gamma_2 \tag{4}$$

where *n* is the unit outward vector situated on the boundary of the problem domain.

Green element formulation (GEM)

The details of GEM formulation have already been fully discussed in earlier papers (Onyejekwe, 1995; 1996) but here some steps are presented to clarify the methodology. GEM is implemented according to:

1. Integral replication of the governing partial differential equation.
2. The presentation and the solution of the integral domain of the boundary or initial value problem on a generic element of the problem domain.
3. Assembly and determination of the field variables at each node.

Before we embark on this discretisation exercise, Step 1 is worthy of comment. We note in passing that in a typical finite element method (FEM), the discrete replica of the governing partial differential equation is arrived at by either the method of **weighted residual** or by variational formulation. Details of both techniques can be read from standard texts on FEM (Reddy, 1984). On the contrary GEM implements this step by the use of the so-called free space Green's function or **the unit response function**. We note that with the weighted residual approach, discrete algebraic equations are obtained by two levels of approximation; one at the partial differential equation by a more direct route which is based on boundary integral theory and which guarantees its second-order convergence. While the BEM finds it difficult to cope with body force terms and non-linear physical properties and heterogeneity because of its "boundary-only" methodology, GEM achieves the flexibility and robustness of FEM by adopting its domain discretisation. The hybrid formulation of GEM actually comes into play in Step 2 and 3 where the element-by-element approach of the FEM is adopted.

The GEM integral replication of Eq. (1) is the same as for BEM and is given as (Onyejekwe, 1996) :

$$-\lambda\phi(x_i) + \int_\Gamma [\phi\nabla G(x,x_i) - G(x,x_i)\nabla\phi(x,x_i)].nds = 0 \tag{5}$$

in which λ is the nodal angle, Γ is the boundary of probe domain, x_i identifies the nodal position of the point source, x is the field point. Since GEM is based on the boundary integral theory, it uses Eq. (5) and the Green's second identity to convert Eq. (1) into its integral form:

$$-\lambda\phi(x_i,t) + \left[\phi\frac{\partial G}{\partial x} - G\frac{\partial \phi}{\partial x}\right]_{x_0}^{x_L} + \int_{x_0}^{x_L} G\left[\left(\frac{\partial\phi}{\partial t} + f(x,t)\right) - \frac{\partial\Phi}{\partial x}\frac{\partial\phi}{\partial x}\right] dx = 0 \tag{6}$$

where f(x,t) is the source or sink term and Φ is the natural logarithm of the diffusivity function.

Eq. (6) can now be simplified to give :

$$\lambda_i\phi(x,t) + \frac{1}{2}\left[H\,[(x_2 - x_i) - H\,(x_i - x_2)]\right]\phi_2 - \left[H\,(x_1 - x_i) - H\,(x_i - x_1)\right]\phi_2 + \frac{1}{2}\left[(|x_2 - x_i| + K)\psi_2 - (|x_1 - x_i| + K)\,\psi_1\right] + \int_{x_1}^{x_2} G\left(\frac{\partial\Phi}{\partial x} + X(\phi)\left[f(x,t) + \frac{\partial\phi}{\partial t}\right]\right)dx = 0 \tag{7}$$

where:

Ψ is the gradient of the dependent variable
X is the reciprocal of the diffusivity function
λ is the nodal angle at the source point and takes the value of unity if the source point is within the problem domain or ½ when located at the boundaries.
$H(x-x_i)$ is the Heaviside function.

In order to relate Eq. (7) to the problem domain, we advanced to Step 2 by dividing the flow domain into elements and approximating the dependent variable and its functions by piece-wise linear basis functions:

$$\Phi \approx \Omega_1(\zeta)\Phi(t) + \Omega_2(\zeta)\Phi(t) \tag{8}$$

$$\frac{\partial\phi}{\partial t} \approx \Omega_1(\zeta)\frac{\partial\phi_1}{\partial t} + \Omega_2(\zeta)\frac{\partial\phi_2}{\partial t} \tag{9}$$

$$f(x,t) \approx \Omega_1(\zeta)f_1(t) + \Omega_2(\zeta)f_2(t) \tag{10}$$

$$X(x,t) \approx \Omega_1(\zeta)\chi_1(t) + \Omega_2(\zeta)\chi_2(t) \tag{11}$$

The accompanying shape functions are defined as:

$$\Omega_1 = \frac{x_{i+1} - x}{l} \qquad \Omega_2 = \frac{x - x_i}{l} \tag{12}$$

where *l* is the length of the element in the problem domain.

Eq. (1) is solved for each element before adding all element contributions. This brings us to Step 3. Summation of all elements in Eq. (7) yields:

$$\sum_{e=1}^{M} \lambda_i^e\phi(x,t) + \frac{1}{2}\left[H\,[(x_2 - x_i) - H\,(x_i - x_2)]\right]\phi_2 - \left[H(x_1 - x_i) - H(x_i - x_1)\right]\phi_2 + \frac{1}{2}\left[(|x_2 - x_i| + K)\,\psi_2 - (|x_1 - x_i| + K)\,\psi_1\right] + \int_{x_1^e}^{x_2^e} G\left(\frac{\partial\Phi}{\partial x} + X(\phi)\left[f(x,t) + \frac{\partial\phi}{\partial t}\right]\right)dx = 0 \tag{13}$$

where *e* represents the e-th element of the problem domain.

We apply Eq. (13) to the two ends of an element whose boundaries ae denoted by $[x_1, x_2]$. When the source is located at Node 1, and Eq. (13) is applied to x_1, we obtain:

$$-\phi_1 + \phi_2 + l_m\psi_1 - (l + l_m)\psi_2 + l^e\int_0^1 (l\zeta + l_m)(-\Phi_1 + \Phi_2)(\Omega_1\psi_1 + \Omega_2\psi_2) + (\Omega_1X_1 + \Omega_2X_2)\left(\Omega_1\frac{\partial\phi_1}{\partial t} + \Omega_2\frac{\partial\phi_2}{\partial t}\right) + (\Omega_1f_1 + \Omega_2f_2)\,dx = 0 \tag{14}$$

where l_m is the maximum length of the problem domain.

Similarly, when the source is located at Node 2, and Eq. (13) applied to x_2, we obtain:

$$\Phi_1 - \Phi_2 - l_m\psi_2 + (l + l_m)\psi_1 + l^e\int_0^1 (l\zeta - l_m)(-\Phi_1 + \Phi_2)(\Omega_1\psi_1 + \Omega_2\psi_2) + (\Omega_1 X_1 + \Omega_2 X_2) \left(\Omega_1\frac{\partial\phi_1}{\partial t} + \Omega_2\frac{\partial\phi_2}{\partial t} + (\Omega_1 f_1 + \Omega_2 f_2)\right) dx = 0 \tag{15}$$

Eqs. (14) and (15) are then combined to give a system of discrete element equations which can be written in the matrix form as:

$$R_{ij}\Phi_j + (L_{ij} - V_{ilj}\Phi_l)\,\psi_j + U_{ijl}X_j\left(\frac{d\Phi_j}{dt} + f_j\right) \tag{16}$$

where R_{ij}, L_{ij}, V_{ilj} and U_{ijl} are matrix coefficients as those defined by Taigbenu and Onyejekwe (1995) and Onyejekwe (1996). Equation (16) is a non-linear matrix equation which requires an iterative process for its solution. Incorporating a time discretisation scheme and Picard's algorithm to the non-linear Eq. (16), we get:

$$\sum_{e=1}^{M}\left[\left(R^e_{il} + \frac{\omega}{\Delta t}U^e_{ijl}X_j^{m+1,k}\right)\phi^{m+1,k+1} - \left(L^e_{ij} + V^e_{ilj}\Phi^{m+1,k}\right)\psi^{m+1,k+1}\right]$$

$$\sum_{e=1}^{M} U^e_{ijl}X_l^{m+1,k+1}\left[\frac{\omega}{\Delta t}\phi_l - (1-\omega)\frac{d\phi^m}{dt} + f\right] \qquad 1 \le \omega \le 2 \tag{17}$$

Example problems and discussion

To validate GEM formulation, we now compare our numerical solution with analytical solutions obtained by Lockington (1997). The example problem considers both recharging and dewatering a homogeneous unconfined shallow sand aquifer bounded by a stream on one side. The aquifer is assumed to be underlain by an impermeable horizontal base considered as a datum level with water table elevation initially at h_0. A hydraulic conductivity of K=20 m/d and a specific yield of S=0.27 are considered. For the recharging process, the water table elevation is initially assumed to be 2.0 m above the datum level for both the stream and the aquifer. The water level elevation in the stream is suddenly increased to 3.0 m as shown in Fig. 1. The closed form solution presented by Lockington (1997) for this case is given by:

$$h = 2 + \left(1 - 0.0217\frac{x}{\sqrt{t}}\right) \tag{18}$$

where x is the horizontal distance and t is the time.

Similarly, for the dewatering process, the same aquifer is assumed with water level elevation initially at 3.0 m above the datum level and then suddenly reduced to 2.0 m as shown in Fig. 2. The closed form solution presented by Lockington (1997) is given by:

$$h = 3 - \left(1 - 0.0136\frac{x}{\sqrt{t}}\right)^{3.9328} \tag{19}$$

Results obtained by GEM and those presented by Lockington (1997) for recharging and dewatering are shown in Figs. 3 and 4 respectively. Figure 3 describes a recharging water table. Note the increase in head with time, and closeness of GEM and Lockington (1997) solutions. The dewatering process as shown in Fig. 4 is the reverse of the simulation described above.

Having performed excellently for the cases treated above, we decided to try GEM formulation on a more demanding problem which involves a two-stream unconfined aquifer system (Fig. 5). For the problem under consideration, the aquifer is exposed to constant and continuous recharge with downstream water-level lowering. The water level behaviour which descends linearly during a 50.0 min period is described by Guillermo and Gabriel (1984) as:

$$h_2(t) = \begin{cases} 10 - 0.1t \ (cm), & 0 \le t \le 50 \text{ min} \\ 5 (cm), & t > 50 \text{ min} \end{cases} \tag{20}$$

For the initial condition, we adopt the analytic expression given by Marino (1973) to describe the groundwater movement in a two-stream unconfined-aquifer system:

$$h^2(x,t_e) = h_1^2 - \frac{(h_1^2 - h_2^2)\,x}{L} + \frac{I_0}{K}(L - x)\,x \tag{21}$$

where h_1 and h_2 are known values of water level at the upstream and downstream boundaries respectively, t_e is the period at the end when equilibrium condition is attained, L is the length of the aquifer,

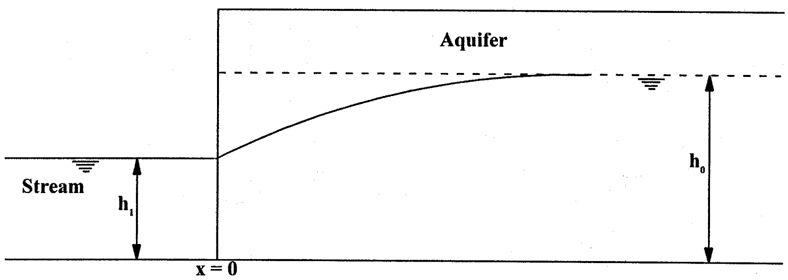


Figure 1
Discharging aquifer

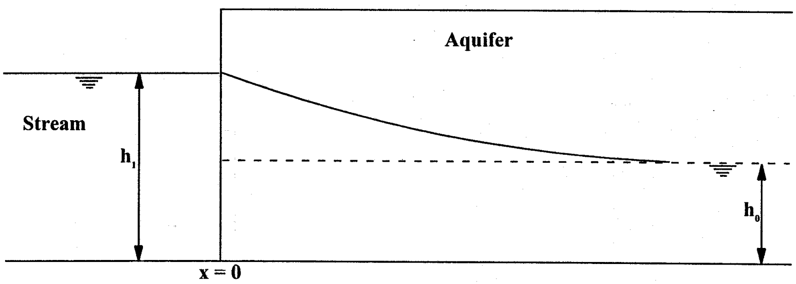
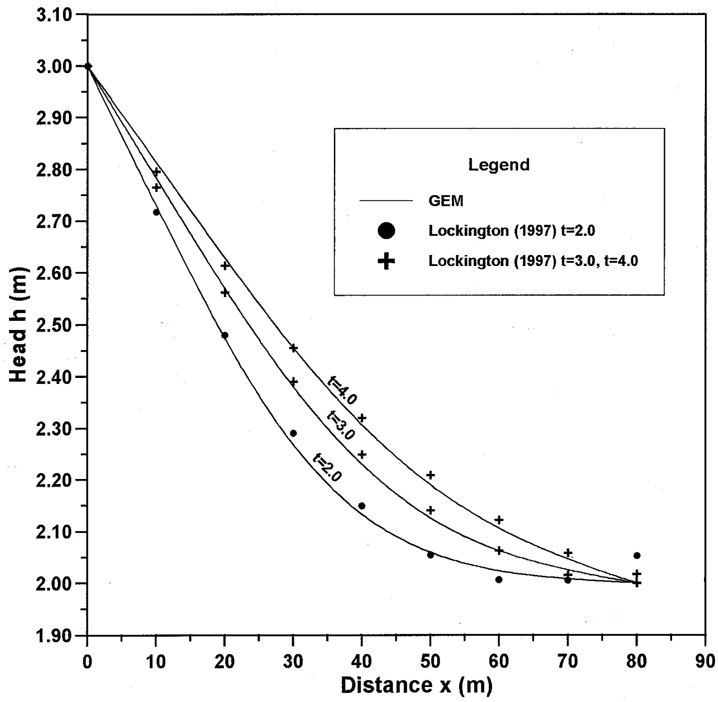
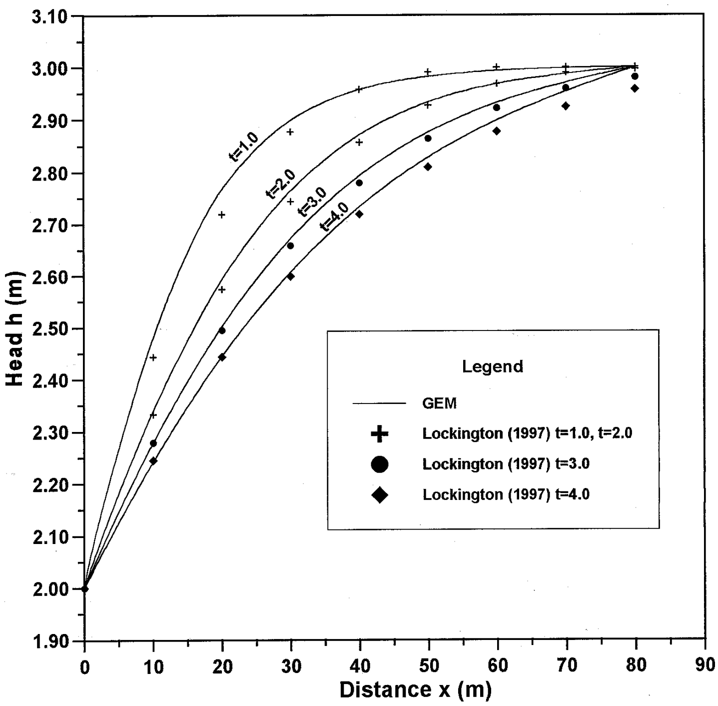


Figure 2
Recharging aquifer





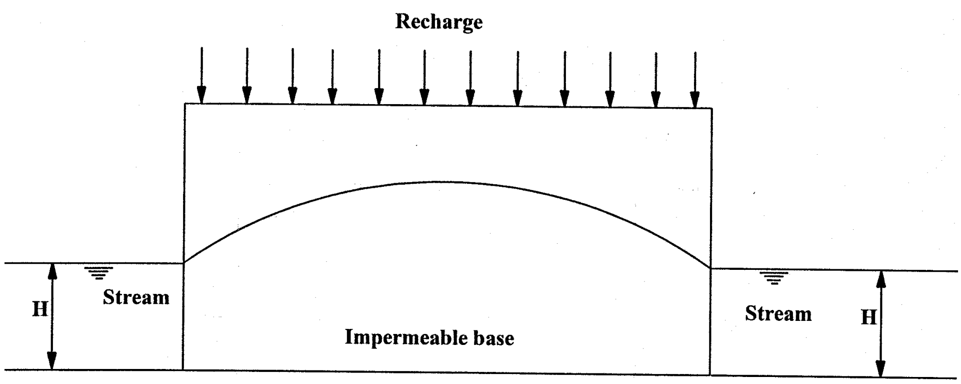


Figure 3 (top)
Variation of head with distance at various times

Figure 4 (middle)
Variation of head with distance at various times

Figure 5 (bottom)
Two-stream aquifer system

I_0 is the infiltration rate and K is the hydraulic conductivity. The initial condition, h(x,0), is obtained when I_0 = 0.

Numerical results obtained from GEM are then compared with analytical solution presented by Guillermo and Gabriel (1984) for different times as shown in Fig. 6. The values of the parameters used are h_1= 30.0 cm, L= 100.0 cm, K= 0.1 cm/min and the specific yield, S= 0.15. It can be seen from Fig. 6 that there is a good agreement between GEM for all the different values and that of Guillermo and Gabriel (1984).

Conclusions

A novel boundary integral solution was developed for determining the water table elevation in an unconfined homogeneous aquifer subjected to recharge and dewatering from a stream as well as fluctuations induced by constant and continuous recharge in a two-stream unconfined-aquifer system. The solution produced excellent results when compared with those available from literature.

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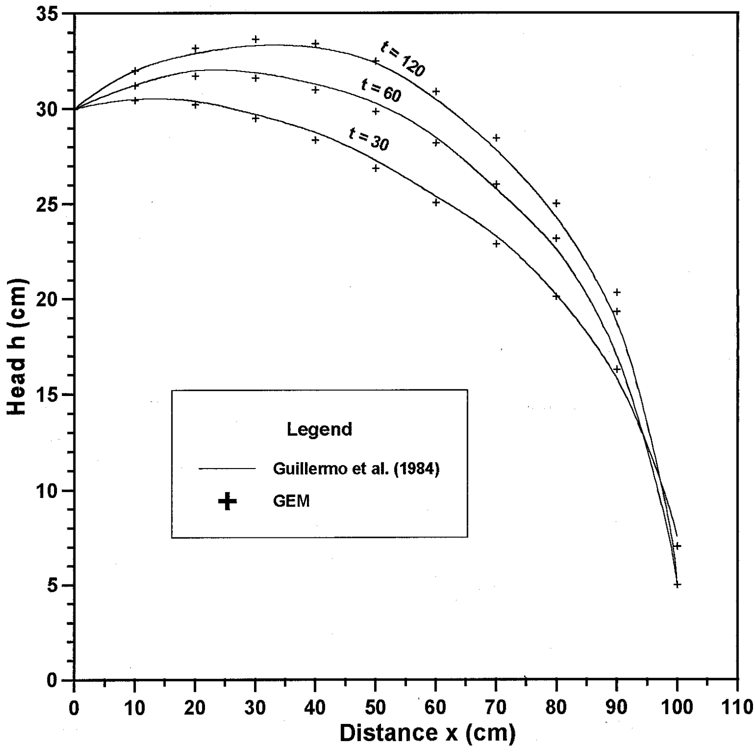

Figure 6
Variation of head with distance at various times